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REGULAR ARTICLE

A generalization of the transmuted Rayleigh distribution

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Abstract: In this paper, we introduce a new family of distributions for modeling positive data. The new distribution arises from the quotient of two independent random variables: transmuted Rayleigh in the numerator, and beta in the denominator. Structural properties of the new distribution are derived, and an application to real data reveals good performance of this new distribution in practice.

Keywords: Kurtosis, Rayleigh distribution, Slash distribution, Skewness, Transmuted distributions

MSC: 60E05, 60E10

1 Introduction

The one-parameter Rayleigh distribution is a continuous probability distribution for modeling positive data. It is named after the English Lord Rayleigh. This distribution plays a key role in modeling lifetime data such as project effort loading modeling, survival and reliability data, theory of communication, physical sciences, technology, diagnostic imaging, and clinical research. Some important references regarding the Rayleigh distribution are (Cliff and Ord, 1975) and (Hirano, 1986). Cliff and Ord (1975) pointed out the remarkable property that the Rayleigh distribution arises as the distribution of the distance between an individual and its nearest neighbor when the spatial pattern is generated by a Poisson process, while (Hirano, 1986) has presented a brief account of the history and properties of this distribution. For further details on the Rayleigh distribution, the reader is referred to (Johnson et al., 1995) and the references cited therein. However, researchers have focused on the generalization of the one-parameter Rayleigh distribution to create a more flexible Rayleigh distribution. A random variable X is said to have a transmuted distribution if its cumulative distribution function (cdf) is given by $F(x) = (1+\lambda)H(x) + \lambda H^2(x)$, where $|\lambda| \leq 1$ and $H(x)$ is the cdf of the baseline distribution. Note that if $\lambda = 0$ we obtain the baseline $H(x)$ distribution of the random variable X. By using this procedure, (Merovci, 2013) studied the two-parameter transmuted Rayleigh ('TR' for short) distribution. A random variable Z is said to have the TR distribution if its cdf and probability density function (pdf) are given, respectively, by

$$F_Z(z;\sigma,\lambda) = \left(1 - e^{-\frac{z^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{z^2}{2\sigma^2}}\right), \tag{1}$$

$$f_Z(z;\sigma,\lambda) = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} \left(1 - \lambda + 2\lambda e^{-\frac{z^2}{2\sigma^2}}\right),$$

where $z > 0$, $\sigma > 0$ and $|\lambda| \leq 1$. We write $Z \sim TR(\sigma,\lambda)$. If $\lambda = 0$ we obtain the Rayleigh distribution. In this short note, we introduce a new generalization of the TR distribution. The main aim of this paper is to introduce a three-parameter Rayleigh family of distributions, which extends the Rayleigh and TR distributions, with the hope that the new distribution may have a âĂŸbetter fitâĂŹ compared to these distributions in certain practical situations. Additionally, we will provide some mathematical properties of the proposed new family of distributions. As we will see later, the formulae related with the new distribution are simple and manageable, and with the use of modern computer resources and its numerical capabilities, the proposed distribution may prove to be an useful addition to the arsenal of applied statisticians in data analysis. In Section 2, we introduce the new three-parameter Rayleigh family of distributions and investigate its properties, including the stochastic representation. Section 3 discusses estimation of model parameters. A simulation study is performed in Section 4. Section 5 provides a real data application for illustrative purposes. Section 6 concludes the paper.

2 Slashed transmuted Rayleigh distribution

2.1 Stochastic representation

We have the following definition.

Definition 1. *A random variable T has the slashed transmuted Rayleigh (STR) distribution if it can be represented as*

$$T = \frac{X}{U^{1/q}}, \tag{2}$$

where $X \sim TR(\sigma,\lambda)$ defined in (1) and $U \sim Uniform(0,1)$ (with X and U independent), $\sigma > 0$, $|\lambda| \leq 1$, and $q > 0$.

Remark 1. *We shall use the notation $T \sim STR(\sigma,\lambda,q)$.*

Proposition 1. *Let $T \sim STR(\sigma,\lambda,q)$. The pdf of T is given by*

$$f_T(t;\sigma,\lambda,q) = \frac{q\sigma^q}{t^{q+1}}\Gamma\left(\frac{q}{2}+1\right)\left[2^{q/2}(1-\lambda)G\left(\frac{t^2}{2\sigma^2},\frac{q}{2}+1,1\right) + \lambda G\left(\frac{t^2}{\sigma^2},\frac{q}{2}+1,1\right)\right], \tag{3}$$

where $t > 0$, $\sigma > 0$, $|\lambda| \leq 1$, $q > 0$, $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}\,du$ is the gamma function, and $G(x;\alpha,\beta) = (\beta^\alpha/\Gamma(\alpha))\int_0^x u^{\alpha-1}e^{-\beta u}\,du$ is the cdf of the gamma distribution.

Proof. Using representation in (2) and the Jacobian method, the pdf associated to T is

$$f_T(t;\sigma,\lambda,q) = \frac{q(1-\lambda)}{\sigma^2}\int_0^1 t w^{q+1} e^{-\frac{t^2w^2}{2\sigma^2}}\,dw + \frac{2q\lambda}{\sigma^2}\int_0^1 t w^{q+1} e^{-\frac{t^2w^2}{\sigma^2}}\,dw.$$

Considering the change of variable $u = t^2w^2/2\sigma^2$ in the first integral and $u = t^2w^2/\sigma^2$ in the second, we obtain

$$f_T(t;\sigma,\lambda,q) = q\sigma^q t^{-(q+1)}\left((1-\lambda)2^{q/2}\int_0^{t^2/(2\sigma^2)} u^{q/2}e^{-u}\,du + \lambda\int_0^{t^2/\sigma^2} u^{q/2}e^{-u}\,du\right).$$

Recognizing the cdf of the gamma distribution, the result follows. □

Remark 2. *When $\sigma = \lambda = q = 1$, the pdf of T reduces to*

$$f_T(t) = \frac{\sqrt{\pi}}{2}t^{-2}G\left(\frac{t^2}{2},\frac{3}{2},1\right), \quad t > 0.$$

Figure 1 shows some pdf shapes of the STR distribution for some parameter values. It is evient that the STR distribution is much more flexible than the TR distribuiton.

2.2 Structural properties

Here, we derive some basic properties of the STR distribution. We have the following proposition.

Proposition 2. *Let $T \sim STR(\sigma,\lambda,q)$. We have that:*

(a) $\lim_{(\lambda,q)\to(0,\infty)} f_T(t;\sigma,\lambda,q) = \frac{t}{\sigma^2}e^{-\frac{t^2}{2\sigma^2}}$.

(b) $\lim_{(\lambda,q)\to(1,\infty)} f_T(t;\sqrt{2}\sigma,\lambda,q) = \frac{t}{\sigma^2}e^{-\frac{t^2}{2\sigma^2}}$.

(c) $\lim_{(\lambda,q)\to(-1,\infty)} f_T(t;(2\beta)^{-1/2},\lambda,q) = 4\beta x e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)$.

(d) $\lim_{\lambda\to 0} f_T(t;\sqrt{\sigma},\lambda,q) = q(2\sigma)^{q/2}\Gamma\left(\frac{q}{2}+1\right)G\left(\frac{t^2}{2\sigma},\frac{q}{2}+1,1\right)$.

(e) $\lim_{\lambda\to 1} f_T(t;\sqrt{2\sigma},\lambda,q) = q(2\sigma)^{q/2}\Gamma\left(\frac{q}{2}+1\right)G\left(\frac{t^2}{2\sigma},\frac{q}{2}+1,1\right)$.

(f)

$$\lim_{\lambda\to -1} f_T(t;(2\beta)^{-1/2},\lambda,q) = \frac{2q}{\lambda^{\frac{q}{2}}t^{q+1}}\Gamma\left(\frac{q}{2}+1\right)\left[G\left(\lambda t^2,\frac{q}{2}+1,1\right) - 2^{-\left(\frac{q}{2}+1\right)}G\left(2\lambda t^2,\frac{q}{2}+1,1\right)\right].$$

(g) $F_T(t;\sigma,\lambda,q) = F_Z(t;\sigma,\lambda) - \frac{t}{q}f_T(t;\sigma,\lambda,q)$, *where $F_Z(\cdot)$ is the cdf given in (1), and $f_T(\cdot)$ is pdf given in (3).*

Proof. Firstly, we have from (2) that the STR distribution reduces to the TR distribution as $q \to \infty$. With this in mind, we immediately have that $\lim_{(\lambda,q)\to(0,\infty)} f_T(t;\sigma,\lambda,q) = \lim_{\lambda\to 0} f_Z(t;\sigma,\lambda) = (t/\sigma^2)e^{-\frac{t^2}{2\sigma^2}}\lim_{\lambda\to 0}(1-\lambda+2\lambda e^{-\frac{t^2}{2\sigma^2}}) = (t/\sigma^2)e^{-\frac{t^2}{2\sigma^2}}$, and so (a) holds. The other ones are obtained in a similar fashion. □

Remark 3. *From Proposition 2, properties 1 and 2 show that the STR distribution tends to the classical Rayleigh distribution when $(\lambda,q)\to(0,\infty)$. Property 3 shows that the STR distribution converges to the exponentiated Rayleigh distribution when $(\lambda,q)\to(-1,\infty)$ and $\sigma = (2\beta)^{-1/2}$. Properties 4 and 5 show that the STR distribution converges to the slashed Rayleigh distribution. Property 6 shows that the distribution STR distribution tends to slashed exponentiated Rayleigh distribution when $\lambda\to -1$.*

We have the following proposition.

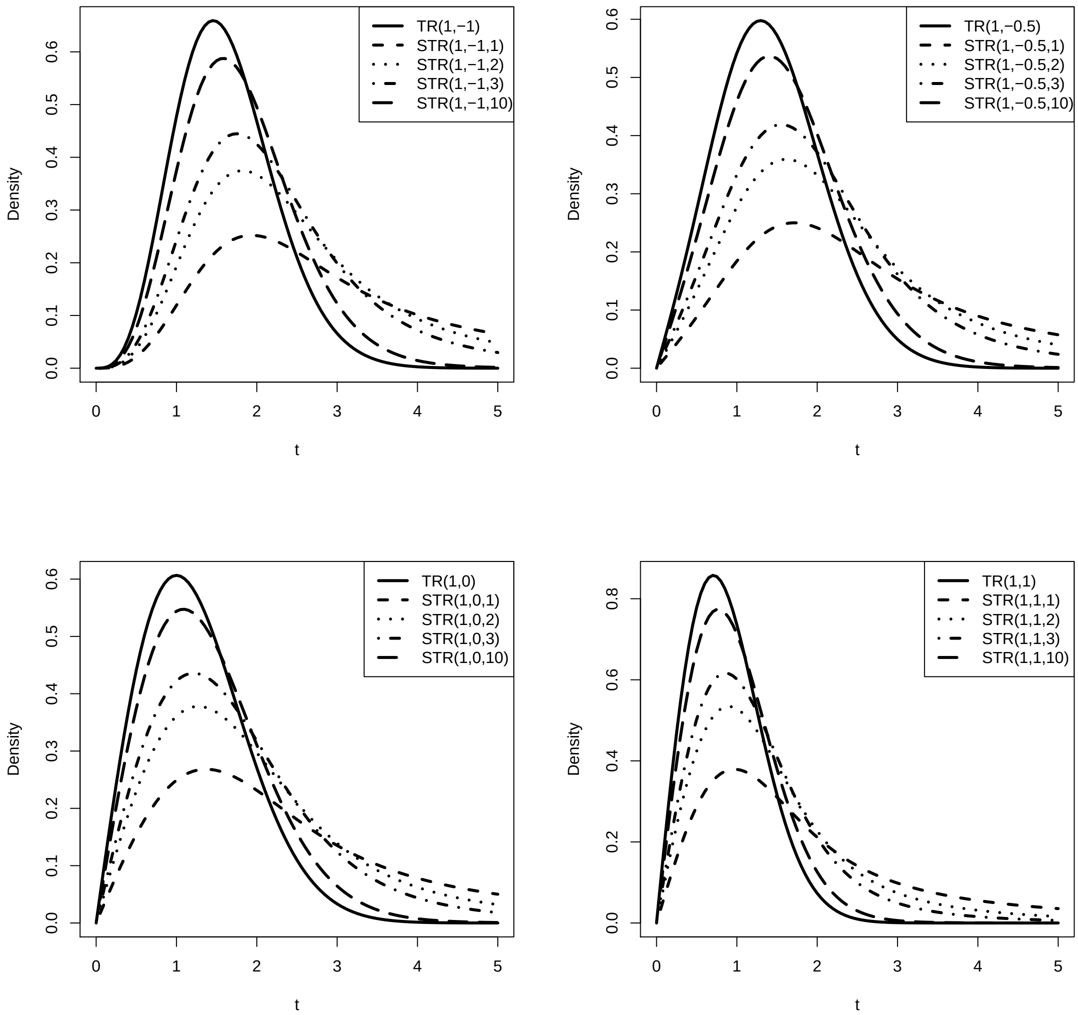

Figure 1: STR density for some parameter values.

Proposition 3. *Let $T \sim STR(\sigma, \lambda, q)$. The r-th moment of T takes the form*

$$E(T^r) = \frac{\sigma^r r q}{2(q-r)}\Gamma\left(\frac{r}{2}\right)(\lambda + 2^{r/2}(1-\lambda)), \qquad q > r, \tag{4}$$

where $r = 1, 2, \ldots$.

Proof. Using the stochastic representation in (2), we have $E\left(W^{-r}\right) = q/(q-r)$ for $q > r$, and $E\left(X^r\right) = \frac{\sigma^r r}{2}\Gamma(r/2)(\lambda + 2^{r/2}(1-\lambda))$ are the moments of the $TR(\sigma, \lambda)$ distribution. □

From (4), we have that

$$\begin{aligned} E(T) &= \frac{\sigma q\sqrt{\pi}}{2(q-1)}(\lambda + \sqrt{2}(1-\lambda)), \quad q > 1, \\ Var(T) &= \sigma^2 q\left(\frac{2-\lambda}{q-2} - \frac{q\pi}{4}(\lambda + \sqrt{2}(1-\lambda))^2\right), \quad q > 2. \end{aligned}$$

In addition, we have that the asymmetry coefficient ($\sqrt{\beta_1}$) and the kurtosis coefficient (β_2) are given, respectively, by

$$\sqrt{\beta_1} = \frac{\frac{3(\lambda+2\sqrt{2}(1-\lambda))}{4(q-3)} - \frac{3(\lambda+\sqrt{2}(1-\lambda))(2-\lambda)q}{2(q-1)(q-2)} + \frac{\pi(\lambda+\sqrt{2}(1-\lambda))^3 q^2}{4(q-1)^3}}{\sqrt{\frac{q}{\pi}}\left(\frac{2-\lambda}{q-2} - \frac{\pi(\lambda+\sqrt{2}(1-\lambda))^2 q}{4(q-1)^2}\right)^{3/2}}, \quad q > 3,$$

$$\beta_2 = \frac{\frac{2(4-3\lambda)}{q-4} - \frac{3\pi(\lambda+\sqrt{2}(1-\lambda))(\lambda+2\sqrt{2}(1-\lambda))q}{2(q-1)(q-3)} + \frac{3\pi(\lambda+\sqrt{2}(1-\lambda))^2(2-\lambda)q^2}{2(q-1)^2(q-2)} - \frac{3\pi^2(\lambda+\sqrt{2}(1-\lambda))^4 q^3}{16(q-1)^4}}{q\left(\frac{2-\lambda}{q-2} - \frac{\pi(\lambda+\sqrt{2}(1-\lambda))^2 q}{4(q-1)^2}\right)^2}, \quad q > 4.$$

When $(\lambda, q) \to (0, \infty)$, the asymmetry and kurtosis tend, respectively, to

$$(\pi - 3)\sqrt{\frac{4\pi}{(4-\pi)^2}}, \quad \frac{32-3\pi^2}{(4-\pi)^2},$$

which are the asymmetry and kurtosis of the Rayleigh distribution. When $q \to \infty$, the asymmetry and kurtosis converge, respectively, to

$$\frac{\frac{3}{4}(\lambda+2\sqrt{2}(1-\lambda)) - \frac{3}{2}(\lambda+\sqrt{2}(1-\lambda))(2-\lambda) + \frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^3}{\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)^{3/2}},$$

and

$$\frac{2(4-3\lambda) - \frac{3\pi}{2}(\lambda+\sqrt{2}(1-\lambda))(\lambda+2\sqrt{2}(1-\lambda)) + \frac{3\pi}{2}(\lambda+\sqrt{2}(1-\lambda))^2(2-\lambda) - \frac{3\pi^2}{16}(\lambda+\sqrt{2}(1-\lambda))^4}{\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)^2},$$

which are the asymmetry and kurtosis of the TR distribution. Figure 2 shows the graphs of the asymmetry and kurtosis of the STR distribution as functions of λ and q. Note that, for a fixed value of λ, the kurtosis of the STR distribution goes to zero as $q \to \infty$, while it has no upper limit as $q \to 4^+$.

3 Parameter estimation

In the following, we consider the method of moments as well as the maximum likelihood method to estimate the parameters σ, λ and q of the STR distribution.

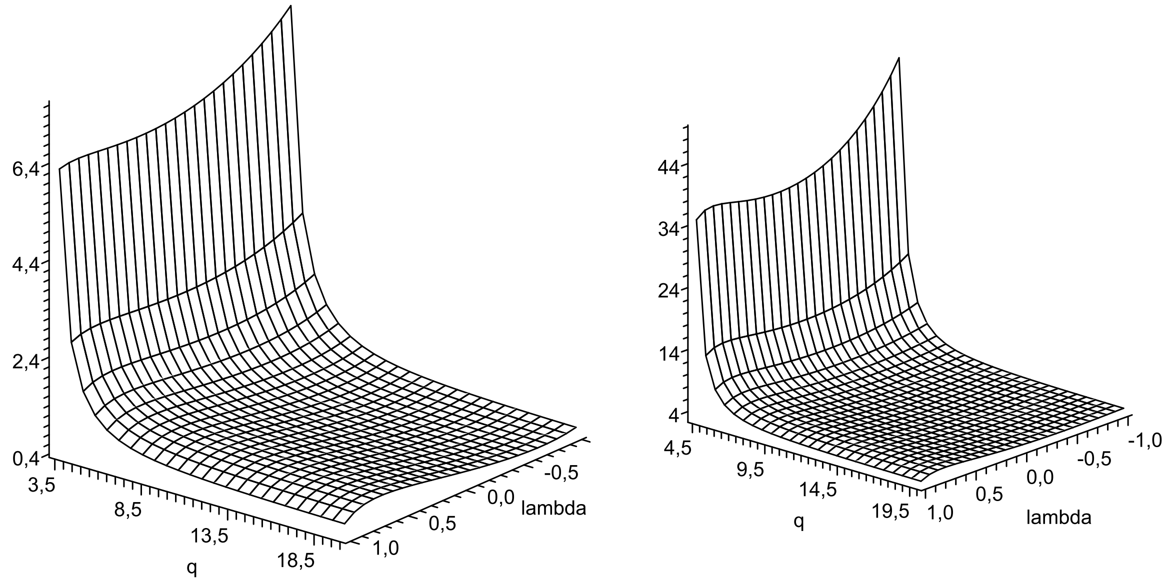

Figure 2: (left) Asymmetry; (right) Kurtosis.

3.1 Moment estimators

We have the following proposition.

Proposition 4. *Let $t_1, \ldots, t_n$ be a sample of size n from the $T \sim STR(\sigma, \lambda, q)$ distribution. The moment estimates of σ, λ and q, for $q > 3$, are*

$$\widetilde{q} = \frac{1}{1 - d_1\sigma}, \quad \widetilde{\sigma} = \frac{d_1 \pm \sqrt{d_1^2 - d_2}}{d_2},$$

$$0 = \left(\frac{d_1}{d_2}\right)^2 \left(\frac{2d_3}{d_2} - 3\right)^2 (d_1^2 - d_2) - \left(\left(\frac{3d_1^2}{d_2} - 2\right) - \frac{d_3}{d_2^2}\left(2d_1^2 - d_2\right)\right)^2, \tag{5}$$

where $d_1 = \frac{\sqrt{\pi}}{2\overline{T}}(\lambda + \sqrt{2}(1-\lambda))$, $d_2 = \frac{2-\lambda}{\overline{T^2}}$, $d_3 = \frac{3\sqrt{\pi}}{4\overline{T^3}}(\lambda + 2\sqrt{2}(1-\lambda))$, and $\overline{T} = (1/n)\sum_{i=1}^n t_i$ is the sample mean, $\overline{T^2} = (1/n)\sum_{i=1}^n t_i^2$, and $\overline{T^3} = (1/n)\sum_{i=1}^n t_i^3$. The equation (5) depends only on the parameter λ, and so it has to be solved numerically to obtain $\widetilde{\lambda}$.

Proof. Using (4), we have that

$$\begin{aligned} E(T) &= \frac{\sqrt{\pi}\sigma q}{2(q-1)}(\lambda + \sqrt{2}(1-\lambda)), \\ E(T^2) &= \frac{\sigma^2 q}{(q-2)}(2-\lambda), \\ E(T^3) &= \frac{3\sqrt{\pi}\sigma^3 q}{4(q-3)}(\lambda + 2\sqrt{2}(1-\lambda)), \end{aligned} \tag{6}$$

and thus replacing $E(T)$ with $\overline{T}$, $E(T^2)$ with $\overline{T^2}$ and $E(T^3)$ with $\overline{T^3}$ in (6), we obtain the moment estimators $(\widetilde{\sigma}, \widetilde{\lambda}, \widetilde{q})$ of (σ, λ, q). $\square$

3.2 Maximum likelihood

Suppose $t_1, t_2, \ldots, t_n$ is a sample of size n from the $STR(\sigma, \lambda, q)$ distribution. The log-likelihood function is

$$\log(L(\sigma,\lambda,q)) = c(\sigma,\lambda,q) - (q+1)\sum_{i=1}^{n}\log(t_i)$$
$$+\sum_{i=1}^{n}\log\left(2^{q/2}(1-\lambda)G\left(\frac{t_i^2}{2\sigma^2},\frac{q}{2}+1,1\right)+\lambda G\left(\frac{t_i^2}{\sigma^2},\frac{q}{2}+1,1\right)\right),$$

where $c(\sigma,\lambda,q) = n\log(q) + nq\log(\sigma) + n\log(\Gamma\left(\frac{q}{2}+1\right))$. Taking the derivatives of the log-likelihood function $\log(L(\sigma,\lambda,q))$ with respect to σ, λ and q, and equating these expressions to zero, we obtain

$$\frac{nq}{\sigma} + \sum_{i=1}^{n}\frac{F_1(t_i)}{F(t_i)} = 0,$$
$$\sum_{i=1}^{n}\frac{F_2(t_i)}{F(t_i)} = 0,$$
$$\frac{n}{q} + n\log(\sigma) + \frac{n}{2}\Psi\left(\frac{q}{2}+1\right) - \sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\frac{F_3(t_i)}{F(t_i)} = 0,$$

where $F(t_i) = 2^{q/2}(1-\lambda)G(t_i^2/(2\sigma^2), q/2+1, 1) + \lambda G(t_i^2/\sigma^2, q/2+1, 1)$, $F_1(t_i) = \frac{d}{d\sigma}F(t_i)$, $F_2(t_i) = \frac{d}{d\lambda}F(t_i)$, $F_3(t_i) = \frac{d}{dq}F(t_i)$, and $\Psi(\cdot)$ is the digamma function. The above equations do not lead to explicit analytical solutions for the maximum likelihood (ML) estimates $\widehat{q}$, $\widehat{\sigma}$ and $\widehat{\lambda}$ of q, σ and λ, respectively. The Newton-Raphson iterative technique could be applied to solve the above likelihood equations and obtain the estimates $\widehat{q}$, $\widehat{\sigma}$ and $\widehat{\lambda}$ numerically. The observed information matrix used for computing confidence intervals for the parameters q, σ and λ, and to compute standard errors (SD) for the estimates $\widehat{q}$, $\widehat{\sigma}$ and $\widehat{\lambda}$ can be determined numerically from standard maximization routines, which now provide the observed information matrix as part of their output; e.g., one can use the R function `optim` to compute the observed information matrix numerically; see R Core Team (2022).

4 Simulation study

In this section, a small Monte Carlo simulation experiment is performed to illustrate the behavior of the ML estimators of the model parameters σ, λ and q. We generate 10,000 random samples of sizes n =100, 200 and 500 from the STR distribution. Random numbers of $T \sim STR(\sigma,\lambda,q)$ can be generated as follows:

1. Generate $U \sim \mathcal{U}(0,1)$, that is, a uniform distribution on the unit interval $(0,1)$.
2. Compute $X = q_R\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda U}}{2\lambda}\right)$, where $q_R(\cdot)$ is the quantile function of the Rayleigh distribution.
3. Compute $T = X/U^{1/q}$.

The evaluation of point estimation was performed based on the following quantities for each sample size: the empirical mean and the empirical standard deviation (eSD). It can be seen from Table 1 that the ML estimates are quite stable and, more important, are close to the true values for the sample sizes considered. Additionally, as the sample size increases, the eSD decreases, as expected.

[htp!]

Table 1: Empirical mean and eSD for the ML estimates.

σ	λ	q	$\widehat{\sigma}$ (eSD)	$\widehat{\lambda}$ (eSD)	$\widehat{q}$ (eSD)
			$n=100$		
1.0	-1.0	1.0	1.044 (0.147)	- 0.933 (0.174)	1.028 (0.145)
		2.0	1.040 (0.103)	- 0.948 (0.115)	2.141 (0.395)
		3.0	1.029 (0.095)	- 0.953 (0.101)	3.357 (1.349)
2.0	0.5	1.0	1.040 (0.177)	0.546 (0.401)	1.023 (0.156)
		2.0	1.032 (0.156)	0.532 (0.365)	2.122 (0.623)
		3.0	1.030 (0.150)	0.535 (0.342)	3.591 (3.038)
3.0	1.0	1.0	2.824 (0.420)	0.770 (0.261)	1.036 (0.162)
		2.0	2.855 (0.384)	0.774 (0.254)	2.249 (0.925)
		3.0	2.911 (0.373)	0.794 (0.241)	4.155 (4.892)
			$n=200$		
σ	λ	q	$\widehat{\sigma}$ (eSD)	$\widehat{\lambda}$ (eSD)	$\widehat{q}$ (eSD)
1.0	-1.0	1.0	1.023 (0.076)	- 0.961 (0.085)	1.014 (0.094)
		2.0	1.013 (0.060)	- 0.972 (0.053)	2.047 (0.240)
		3.0	1.016 (0.058)	- 0.970 (0.055)	3.155 (0.503)
2.0	0.5	1.0	2.043 (0.275)	0.538 (0.364)	1.006 (0.107)
		2.0	2.026 (0.265)	0.513 (0.353)	2.034 (0.323)
		3.0	2.022 (0.264)	0.496 (0.353)	3.130 (0.954)
3.0	1.0	1.0	2.859 (0.331)	0.799 (0.253)	1.028 (0.109)
		2.0	2.847 (0.308)	0.794 (0.248)	2.107 (0.327)
		3.0	2.871 (0.294)	0.806 (0.234)	3.589 (5.585)
			$n=500$		
σ	λ	q	$\widehat{\sigma}$ (eSD)	$\widehat{\lambda}$ (eSD)	$\widehat{q}$ (eSD)
1.0	-1.0	1.0	1.010 (0.044)	- 0.978 (0.040)	1.006 (0.061)
		2.0	1.008 (0.038)	- 0.983 (0.031)	2.029 (0.147)
		3.0	1.007 (0.034)	- 0.982 (0.030)	3.039 (0.280)
2.0	0.0	1.0	2.016 (0.232)	0.509 (0.334)	1.000 (0.064)
		2.0	2.035 (0.216)	0.546 (0.306)	1.996 (0.181)
		3.0	2.020 (0.217)	0.520 (0.298)	2.996 (0.381)
3.0	1.0	1.0	2.831 (0.279)	0.792 (0.252)	1.011 (0.068)
		2.0	2.855 (0.255)	0.819 (0.243)	2.057 (0.196)
		3.0	2.868 (0.248)	0.823 (0.237)	3.188 (0.455)

5 Real data illustration

Here, we consider the real dataset in (Devore, 2012), which corresponds to the lifetime of Microdrills (number of holes that a drill machines before it breaks). The sample size is $n = 50$. The moment estimates of the model parameters are $\widetilde{\sigma} = 59.206$, $\widetilde{\lambda} = -0.248$ and $\widetilde{q} = 3.007$. Now, using these estimates as starting values to compute the ML estimates, we obtain the ML estimates (standard errors (SD) between parentheses) as given in Table 2 for the TR and STR distributions. We consider the `optim` function of the R program (see ("R, 2022)) to maximize the log-likelihood function. The Akaike Information Criterion (AIC) introduced by (Akaike, 1974) and Bayesian Information Criterion (BIC) proposed by (Schwarz, 1978) are also provided in Table 2. We have that AIC $= 2k - 2\widehat{\ell}$, and BIC $= k\ln(n) - 2\widehat{\ell}$, where $\widehat{\ell}$ denotes the maximized value of the log-likelihood function, and k is the total number of parameters. From Table 2, it is evident that STR distribution outperforms the TR distribution according to the AIC and BIC values. In order to assess if the model is appropriate, the histogram of the dataset and plots of the fitted TR and STR distributions are displayed in Figure 3. The plot indicates that the STR distribution yields the best fit and hence can be an adequate model for these data. In summary, the proposed model is a good alternative to the TR available in the statistical literature

[h!]

Table 2: ML estimates (SD) of the TR and STR model parameters.

Model	$\widehat{\sigma}$	$\widehat{\lambda}$	$\widehat{q}$	AIC	BIC
TR	125.893 (11.333)	0.698 (0.174)	-	581.9892	585.8132
STR	48.406 (11.307)	-0.271 (0.533)	2.092 (0.566)	570.9134	576.6495

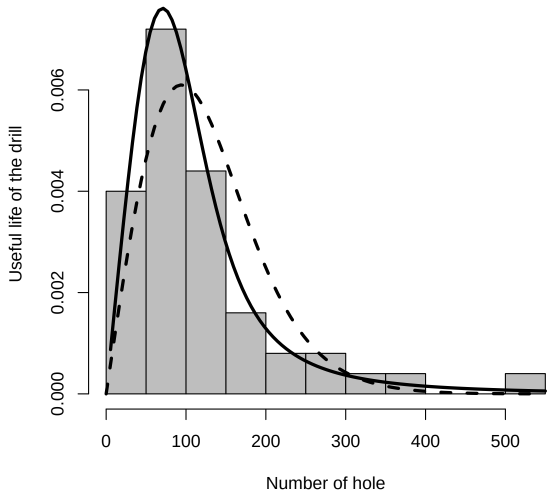

Figure 3: Histogram and fitted models: STR (solid line) and TR (dashed line).

6 Concluding remarks

We introduce a new three-parameter distribution, called the slashed transmuted Rayleigh distribution (STR), and study some of its general structural properties. The model generalizes the transmuted Rayleigh family of distributions and the Rayleigh distribution. This generalization can be used for modeling positive data with high kurtosis. The model parameters are estimated by maximum likelihood and method of moments. A real data illustration reveals that the proposed model can be very useful in practical scenarios.

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