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# Specification and Identification in Spatial Econometric Models

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**Abstract**

This study addresses the problem of using isomorphic expressions to specify and identify spatial econometric models. It then studies the selection of a dynamic process underlying a (multiple) time series; use of a flexible discount function reveals itself to be a precious aid in selecting an appropriate specification.

*Keywords:* discount function, dynamic process, isomorphism, misspecification.

*JEL Classification:* C4, C5, R1

*AMS Classification*: 91B72, 93D25, 91G70, 37A35

## Especificación e Identificación en Modelos Econométricos Espaciales

**Resumen**

Este estudio trata el problema de usar expresiones isomórficas para especificar e identificar modelos econométricos. También se estudia la selección de un proceso dinámico subyacente en una serie temporal (múltiple); la utilización de una función de descuento flexible resulta de gran utilidad en la selección de la especificación apropiada.

*Palabras clave:* función de descuento, proceso dinámico, isomorfismo, mala especificación.

*Clasificación JEL:* C4, C5, R1

*Clasificación AMS*: 91B72, 93D25, 91G70, 37A35

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## 1. Introduction

Spatial econometrics addresses not often specification and identification problems as such; $\rho\mathbf{W}$ linear models, and $\beta$-convergence ones are galore.

This study treats two of the problems mentioned, on the one hand by revealing some isomorphisms, on the other hand by investigation dynamic misspecifications (sections 2 and 3). General conclusions and references follow as usual.

## 2. Some isomorphisms

Very often the search for isomorphisms suggests new solutions to specification problems; two examples will be exposed hereafter.

### 2.1 Consistent spatial modeling

In Paelinck, Das Gupta and Kelekar (2010) it has been shown how the topological structure of the set of spatial units can assist in identifying certain parameters; it will be shown here how this idea can be applied to a complete spatial econometric model.

Take the extended SAR model

$$\mathbf{y} = \mathbf{A}\mathbf{y} + \mathbf{X}\mathbf{b} + \boldsymbol{\varepsilon} \qquad [1]$$

The first fact to be noted is that the $y$ variables have all the same definition, GRP, for instance, contrary to the classical non-spatial model $\mathbf{A}\mathbf{y} + \mathbf{X}\mathbf{b} = \boldsymbol{\varepsilon}$, where the vector $\mathbf{y}$ consists of different variables. A second fact is that model [1] is isomorphic to the classical input-output model

$$\mathbf{y} = \mathbf{A}\mathbf{y} + \mathbf{f} \qquad [2]$$

where $\mathbf{A}$ is the input coefficients matrix, $\mathbf{f}$ the final demand vector.

The difficulty with the spatial econometric model, compared to the input-output one, is that the input coefficients are known from statistical observation, which is not the case of the $\mathbf{A}$ matrix in model [1]; for instance, from equation [2] total relative inputs can be computed as $\mathbf{a}' = \mathbf{i}'\mathbf{A}$ ; this leads to the following suggestion for the spatial econometric model.

The assumption is that total received impacts by a spatial unit i ($a_i$ from vector a' above) are a function of its production level, the higher that level, the higher the impacts; this allows for cross influences of receiving and impacting regions, the latter resulting from the first right-hand side of equation [1]. The proposed function is

$$a_i = 1 - e^{ay(i)} \qquad [3]$$

where $y_i$ is the production level of region $i$; of note is that the exponent in [3] compensates dimensionally. The coefficient $a$ is to be estimated jointly with the coefficients of matrix **A**.

Model (1) will now be applied to the contiguity data of Table 1 (in fact, a starting point for a matrix **W**), and Gross Regional Product data ($10^5$ Euros of 2000) of Table 2, both for Belgium.

Table 1

**Contiguity degrees of the Belgian provinces**

| | A | BW | VB | OV | WV | LIM | H | N | LU | LIE | BC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 0 | 2 | 1 | 1 | 2 | 1 | 2 | 3 | 3 | 2 | 2 |
| BW | 2 | 0 | 1 | 2 | 2 | 2 | 1 | 1 | 2 | 1 | 2 |
| VB | 1 | 1 | 0 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 1 |
| OV | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 2 | 3 | 2 | 2 |
| WV | 2 | 2 | 2 | 1 | 0 | 3 | 1 | 2 | 3 | 3 | 3 |
| LIM | 1 | 2 | 1 | 2 | 3 | 0 | 2 | 2 | 2 | 1 | 2 |
| H | 2 | 1 | 1 | 1 | 1 | 2 | 0 | 1 | 2 | 2 | 2 |
| N | 3 | 1 | 2 | 2 | 2 | 2 | 1 | 0 | 1 | 1 | 3 |
| LU | 3 | 2 | 2 | 3 | 3 | 2 | 2 | 1 | 0 | 1 | 3 |
| LIE | 2 | 1 | 1 | 2 | 3 | 1 | 2 | 1 | 1 | 0 | 2 |
| BC | 2 | 2 | 1 | 2 | 3 | 2 | 2 | 3 | 3 | 2 | 0 |

Abbreviations in ascending alphabetical order: A, Antwerpen; BW, Brabant Wallon; BC, Brussels Capital; H, Hainaut; LIM, Limburg; LIE, Liège; LU, Luxembourg; N, Namur; OV, Oost Vlaanderen; VB, Vlaams Brabant; WV, West Vlaanderen.

Table 2

**Gross Regional Products of the Belgian spatial units, 1995**

| Units | A | BW | VB | OV | WV | LIM | H | N | LU | LIE | BC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Values | 416028 | 62919 | 211584 | 255118 | 226222 | 143460 | 191433 | 64851 | 37976 | 173063 | 424381 |

The spatial units have then been split into three categories: high (A, B, C), medium, and low (BW, N, LU) GRPs, with 11 $a_{ri}$ coefficients: $a_1, a_2, a_3$: zero order contiguity (own) impact; $a_4, a_5, a_6$ first order; $a_7, a_8, a_9$: second order; $a_{10}, a_{11}, a_{12}$: third order. The so created mixed contiguity order has been translated into the matrix **A**, allowing to set up the 11 constraints [3].

As the system is underdetermined, and moreover might be partially inconsistent, the estimation solution was sought along the following lines: the sum of the absolute differences between the values of Table 2, and values guaranteeing consistency, was minimized (the method so chosen discarding possible outliers), which changed the coefficient of OV to .7080, the one of LIM to .6901, and the one of LIE to .7259; this lead, under conditions [3], to the coefficients of Tables 3 and 4, the value of coefficient $\alpha$ being 6.2550E-6

Table 3

**Values of equation (3)**

| Units | A | BW | VB | OV | WV | LIM | H | N | LU | LIE | BC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Val. | .9257 | .3251 | .7335 | .7970 | .7568 | .5921 | .6977 | .3332 | .2113 | .6610 | .9294 |

Table 4

$a_{ri}$ **coefficients**

| Coef. | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ | $\alpha_8$ | $\alpha_9$ | $\alpha_{10}$ | $\alpha_{11}$ | $\alpha_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Val. | .0954 | .0886 | .0640 | .0837 | .0699 | .0651 | .0790 | .0520 | 0 | .0921 | .0802 | .0042 |

The impact of exogenous variables could then be computed from $\mathbf{y^*} = \mathbf{(I - A)y}$ ; **X** from equation [1] was constructed as a binary three sector (agriculture, industry, services) matrix (Table 5).

Table 5

**The X matrix**

| Units\Sector | A | I | S |
|---|---|---|---|
| A | 0 | 1 | 1 |
| BW | 1 | 0 | 0 |
| VB | 1 | 1 | 1 |
| OV | 1 | 1 | 0 |
| WV | 1 | 0 | 1 |
| LIM | 1 | 0 | 0 |
| H | 0 | 1 | 0 |
| N | 1 | 0 | 1 |
| LU | 1 | 0 | 0 |
| LIE | 0 | 1 | 0 |
| BC | 0 | 1 | 1 |

The result (with OLS) was vector **b**=[-73161; 109175; 106678; constant=-11687]' with $R^2$= .6807, significant at the 5% level. The coefficients have admissible signs, agriculture depressing regional products; given the binary construction of **X**, significant coefficients were hardly to be expected.

Now, a very curious fact was observed: the correlation between **y** of equation [1], and **y*** as defined above, is extremely high ($R^2$ = .9968), and the regression of **y** on **y*** gives a slope of .9939 and a constant of 151123; this means that if one takes **y** as the regressand rather than **y*** the difference is approximately only a constant. Indeed one has

$$\mathbf{y} = \mathbf{y}^* + c\mathbf{i} = \mathbf{Xb} + c\mathbf{i} \qquad [4]$$

The regression of **y** on **X** results indeed in a slightly higher $R^2$ (.7028) and the parameters (-67785; 115417; 108489; constant=131475) show indeed slopes of the same order of magnitude as above with a significant higher constant. This means that the results of Thompson and Mattila (1959) obtained without any interregional feedback, might in fact have been the reduced form of [1], so unwittingly taking into account spatial interdependencies.

How about forecasting? This should be performed in an iterative way, given that **A** depends on **y** via equation [3].

Concluding it could be said that a common rule of certain econometric models –to wit input-output ones– can be an important help, together with topological information (see also Paelinck and Klaassen, 1979, in particular pp. 8-9 and footnote 21), in identifying and estimating a multi-regional model.

### 2.2 Lotka-Volterra systems as generalized logistics

Let us first introduce generalized Lotka-Volterra systems (GLVSs); a generalized GLVS can be written in matrix-vector notation as

$$\dot{\mathbf{u}} = \hat{\mathbf{u}}(\mathbf{A}\mathbf{u} + \mathbf{a}) \qquad [5]$$

where **u** is a column-vector of (endogenous) variables, **û** its diagonal matrix version, **A** a square matrix, and **a** is a column-vector of fixed coefficients; the •-notation denotes the time derivative, $\partial / \partial t$.

Given equation [5], the variables **u** describe a time path that can take all the characteristics of general continuous dynamic processes (e.g., convergence, divergence, limit circles; see Braun, 1975, §4.9; Gandolfo, 1996, in particular §24.4; Peschel and Mende, 1986). A sufficient condition can be derived (Griffith and Paelinck, 2009, Chapter 12) for equation [5] to converge to its focus, $-\mathbf{A}^{-1}\mathbf{a}$, by constructing a Lyapunov-function (Hahn, 1963).

From the differential specification of the classical logistic

$$\dot{u} = a^{*}\hat{u}(1 + \lambda u) \qquad [6]$$

one can derive its generalization

$$\dot{\mathbf{u}} = \hat{\mathbf{u}}\mathbf{A}^{*}(\mathbf{i} + \hat{\boldsymbol{\lambda}}\mathbf{u}) \qquad [7]$$

In expression [6] $-\lambda^{-1}$ is the value of the asymptote, to be generalized to the elements of matrix $\lambda$ in equation [7].

The equivalence of [5] and [7] can be shown as follows. Define

$$\mathbf{A}^{*}\mathbf{i} \triangleq \mathbf{a} \qquad [8a]$$

and

$$\mathbf{A}\hat{\boldsymbol{\lambda}} \triangleq \mathbf{A}^{*} \qquad [8b]$$

**i** being the unit column-vector**.** This implies

$$\mathbf{u} = \mathbf{A}^{*}\left(\mathbf{i} + \hat{\boldsymbol{\lambda}}^{-1}\mathbf{u}\right) \qquad [9]$$

**-λ** being the asymptotic vector**.** It is indeed equal to the classical equilibrium vector (if the system converges) $\mathbf{-A}^{-1}\mathbf{a}$ as

$$\mathbf{a} = \mathbf{A}^{*}\mathbf{i} = \mathbf{A}\boldsymbol{\lambda} \qquad [10a]$$

hence

$$\mathbf{-A}^{-1}\mathbf{a} = \mathbf{-A}^{-1}\mathbf{A}\boldsymbol{\lambda} = \mathbf{-}\boldsymbol{\lambda} \qquad [10b]$$

which implies that no matrix inversion is necessary to compute the possible focal point.

These developments will now be applied to a four variable case borrowed from Griffith and Paelinck, 2011, chapter 16; Table 6 presents the data. The case was about a two-region two-sector model.

Table 6

**Logistic data**

| $R_S$\Par. | 1 | 2 | 3 | 4 | Ct |
|---|---|---|---|---|---|
| 1.1 | -6.1627 | 44.1253 | -6.6102 | 2.0707 | 3.3528 |
| 1.2 | 11.7884 | 3.3036 | 2.5080 | -3.8460 | -6.9309 |
| 2.1 | 9.9394 | 4.1536 | -7.9713 | -10.9558 | 2.2372 |
| 2.2 | -13.3512 | -8.3467 | 3.4470 | 6.1802 | 6.0663 |

Application of equations [10] lead to the vector **λ**=(.5098,5214; .5023,5026); the figures are shares of sectoral data per spatial unit, so the possible equilibrium values are fifty-fifty deals. It should be noted that the figures have originally been computed for an integral logistic specification. About convergence it can be seen that tr**A** = -4.6501, so that sufficiency conditions for convergence could be satisfied (they are that the real parts of all the eigenvalues should be strictly negative), but it has to be reminded that those conditions are only sufficient.

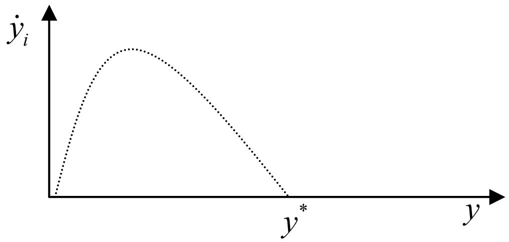
One more interesting point is that equation [7] can be easily generalized to higher order polynomials, e.g. the following quadratic specification between parentheses (see von Bertalanffy, 1973, pp. 60 a.f.)

$$\dot{\mathbf{u}} = \hat{\mathbf{u}}\mathbf{A}^{*}(\mathbf{i} + \hat{\boldsymbol{\lambda}}\mathbf{u} - \hat{\boldsymbol{\mu}}\hat{\mathbf{u}}\mathbf{u}) \qquad [11]$$

The change of signs has been introduced to have an economically meaningful (non-negative) range for the vector **y**, which is the range 0, $\mathbf{y}_i^* = \left[0, \mu_i / 2\lambda_i \left(1 + \sqrt{1} + 4\mu_i / \lambda_i^2\right)\right]$; Figure 1 shows the phase diagram.

Figure 1

**Phase diagram**




In conclusion it should be mentioned that a study of the family links between spatial econometric specifications is a valuable topic to be envisaged for further study; an intrusion into general system theory could also be advocated.

## 3. Space- and time misspecifications

As spatial econometrics *is* about economic behavior, special attention has to be given to the *specification* of the underlying model, the first moments of some distributions, so to speak; this important point will be introduced at first, so as to set the stage for this study.

The standard model in spatial econometrics can be said to be the spatial lag model

$$\mathbf{y} = \rho \mathbf{W} \mathbf{y} + \boldsymbol{\varepsilon} \qquad [12]$$

with **W** a spatial weight matrix, and ρ a spatial lag parameter; it is in fact a (time) static model. This raises the question of how a static model is generated from inherently dynamic (causal; on their links see Heyde, 1957 and Casti, 1997, pp. 189 a.f.) behavior. As an example to illustrate this point, assume the following simple one-equation one-lag model

$$y_t = ax_{t\text{-}1} + \varepsilon_t \qquad [13]$$

and assume further that observations are only available over a double period, so

$$y_t^* = y_{t+1} + y_t \qquad [14.a]$$

$$y_t^* = ax_t^* + a(x_{t-1} - x_{t+1}) + (\varepsilon_t + \varepsilon_{t+1}) \qquad [14.b]$$

where $x_t^* = x_{t\text{-}1} + x_t$; the model is supposed to be fitted as a static one. If the error term has the usual properties, its sum would have zero expectation and double variance (on the influence of spatial aggregation on the error terms, see Paelinck, 2002; and on the influence of dynamic misspecification, Balestra, 1982); but there is a residual term in the $x$-variable, and assumptions should be made about that two-period difference. It could be a constant, or a constant plus a stochastic term; there could be a constant rate of growth, a random walk; anyhow, the dynamics cannot be ignored and has to be specified. Maybe the problem could be neglected if the aggregation process runs like

$$y_t^* = \lambda y_{t+1} + (1-\lambda) y_t \qquad [15]$$

with λ close to 1, in which case the second term of the right hand side of [14.b] turns into $a(1-\lambda)(x_{t-1} - x_{t+1})$ which might produce only a negligible bias in the estimation of parameter $a$. Anyhow, caution is advocated, and this is the reason why next section will be concerned more in detail with space-time specifications.

As to aggregation over space, it has been treated extensively in Paelinck (2002) referring to a problem studied in particular by spatial statisticians as the "Modifiable Areal Unit Problem" (MAUP), the possible use of territorial units of different sizes. In a genuine econometric spirit, this can be treated as a spatial aggregation problem, producing some disturbing consequences for the spatial econometrician. One of these consequences may be summarized as follows: "The important result is that in general econometric aggregation, if only one *macro*-aggregate is considered, *just one* parameter bias is present in the macro-model; in *meso*-aggregation, as it took place here, *every meso-area* has its own specific aggregation bias, which leads to parameter variability between *meso-areas*, and this might result, in econometric estimation, in some sort of (biased) average value, depending on the characteristics of the sample being investigated and the particular spatial aggregation specification".

In larger models the implicit bias will be even more complex; moreover, the stochastic terms of a model will reveal heteroscedasticity and spatial autocorrelation under very general conditions. Of note here is that resulting conclusions impose the use of appropriate specifications adapted to each problem at hand; a possible technique for achieving this end is that of composite parameters —at least when the number of degrees of freedom is adequate—in order to take account of the *specific* bias inherent in each meso-economic spatial unit included in a cross-section analysis (Ancot et al., 1978). But then, how to sort out spatial heterogeneity and spatial bias? Recently filtering data for observational errors, and then for spatial aggregation bias, was proposed in Griffith and Paelinck (2010, Part 2, Chapter 18); the method was applied to a series with maximal spatial complexity, after which complexity was reduced by two thirds, and a simple linear model could be fitted to the filtered data.

### 3.1 Specifications

Several model specifications will be analyzed hereafter from the aggregation point-of-view.

#### 3.1.1 Only exogenous variables

Starting with a one exogenous variable model like in equation [13], assume that the true specification is

$$y_t = \lambda a x_{t-1} + (1-\lambda) a x_{t-2} \tag{16}$$

which means a shared overlap of the impulse over two partial periods. Assuming $x_t = a x_{t-1}$, there comes

$$y_t = a\left[\lambda + \alpha^{-1}(1-\lambda)\right] x_{t-1} \tag{17}$$

over- or under-estimating $a$ depending on the sign and magnitude of α if model [17] is substituted for model [16]. If a static model ($x_t$ as exogenous variable) is used, the result would be

$$y_t = \alpha^{-1}\rho\left[\lambda + \alpha^{-1}(1-\lambda)\right] x_t \tag{18}$$

again or a constant ratio $\rho$ between successive values of the exogenous variable, leading to the same conclusions as above. $\alpha$ being known, $a$ can only be identified if λ is also known, which generally is not the case.

#### 3.1.2 Dynamic processes

Going back to the problem raised around equation [13], assume the true model to be

$$y_t = a y_{t\text{-}1} + b \tag{19}$$

and again two subsequent periods to be aggregated; there comes

$$y_{t+1} + y_t = a\left(y_{t\text{-}1} + y_{t\text{-}2}\right) + a\left(y_t + y_{t\text{-}2}\right) + 2b \tag{20a}$$

$$y_{t+1} + y_t = a^2\left(y_{t\text{-}1} + y_{t\text{-}2}\right) + 2\left(1+a\right)b \tag{20b}$$

In principle $a$ and b are identifiable if knowledge is available about process [19] and the aggregation procedure, but as knowledge about [19] is rather dubious, only result [20b] can be obtained.

If now the underlying process is

$$y_t = ay_{t\text{-}1} + a^* y_{t\text{-}2} + b \qquad [21]$$

and one estimates a one-lag model, $y_{t\text{-}2}$ can be substituted by the instant ratio $r_v = y_{t\text{-}1}/y_{t\text{-}2}$, leading up to

$$y_t = \left(a + r_v^{-1} a^*\right) y_{t\text{-}1} + b \qquad [22]$$

but then the coefficient obtained for the lagged variable (in model [21] $y_{t\text{-}1}$) will be variable over the estimation period as a result of varying $r_v$, and anyway both coefficients $a$ and $a^*$ could not be identified.

### *3.1.3. Multiple variable case*

Here space and the **W** matrix can be reintroduced, generalizing [12] to

$$y_t = \rho \mathbf{W} y_{t\text{-}1} + \rho^* \mathbf{W} y_{t\text{-}2} + \mathbf{b} \qquad [23]$$

and estimating only a one-lag model, one obtains

$$\mathbf{y_t} = \left(\rho \mathbf{W} + \rho^* \mathbf{W}\hat{\mathbf{r}}^{-1}\right) \mathbf{y}_{t\text{-}1} + \mathbf{b} \qquad [24]$$

with $\hat{\mathbf{r}}^{-1}$ the diagonal matrix generalizing $r_v$ of equation [22]. Here the static case can be envisaged, leading up to

$$\mathbf{y}_t = \hat{\mathbf{r}}^{*-1}\left(\rho \mathbf{W} + \rho^* \mathbf{W}\hat{\mathbf{r}}^{-1}\right) \mathbf{y}_t + \mathbf{b} \qquad [25]$$

with an obvious interpretation of $\hat{r}^{*-1}$, but in the resulting condensed specification

$$\mathbf{y}_t = \rho^{**} \mathbf{W} \mathbf{y}_t + \mathbf{b} \qquad [26]$$

$\rho^{**}$ is not identifiable for row-normalized **W** (Paelinck, Das Gupta and Kelekar, 2010).

## 3.2 Simulations

In order to get some insight into the order of magnitude of empirical biases, the coefficients of models [22] and [24] have been estimated, starting from simulated underlying models [21] and [23].

### *3.2.1 Models [21]-[22]*

The true model [21] has been numerically specified as

$$y_t = 1.1 y_{t\text{-}1} - 0.15 y_{t\text{-}2} + 1 \qquad [27]$$

and simulated with starting points (10, 11), Table 7 hereafter listing the ten next figures. Of note is the fact that a negative coefficient has been chosen for the second order lag.

Table 7

**Results of simulating equation [27]**

| $y_t$ | Values |
|---|---|
| 1 | 11.60 |
| 2 | 12.11 |
| 3 | 12.58 |
| 4 | 13.02 |
| 5 | 13.44 |
| 6 | 13.83 |
| 7 | 14.20 |
| 8 | 14.54 |
| 9 | 14.87 |
| 10 | 15.17 |

Process [27] can be shown to be convergent to the value 20.

Equation [22] was reduced to just a first order lag process, and its parameters estimated from the data of Table 7 by SDLS (Griffith and Paelinck, 2011, Part 2, Chapter 12); as progressively less data for the past are available, reduced potentials were computed, but with (equally spaced) linearly decreasing weights, down to .1 (sometimes full past data are available: Agarwal, 2009). This produced a lag parameter of .9396 and a constant equal to 1.2022, which corresponds to the result of equation [22] for $a^* < 0$ and $r_v > 0$.

In *S*ection 3.4 the analysis will be continued.

### *3.2.2 Models [23]-[24]*

To implement model [23] the **W** matrix for two regions has been chosen as in Table 8.

Table 8

**W matrix**

| | |
|---|---|
| .5 | .5 |
| .1 | .9 |

This resulted in the following equations, with $\rho = .8$ and $\rho^* = .25$ and constants .3 and .5:

$$y_{1t} = .40 y_{1,t-1} + .40 y_{2,t-1} + .125 y_{1,t-2} + .125 y_{2,t-2} + .3 \qquad [28a]$$

$$y_{2t} = .08 y_{1,t-1} + .72 y_{2,t-1} + .025 y_{1,t-2} + .225 y_{2,t-2} + .5 \qquad [28b]$$

The system has two real negative roots, and two complex conjugate roots outside the unit circle, so is not convergent.

Simulations with stating point (10,11;5,5.2) produced Table 9.

Table 9

**Results of simulating equations [16]**

| $y_{1t}$, $y_{2t}$ | Values of $y_{1t}$ | Values of $y_{2t}$ |
|---|---|---|
| 1 | 8.66 | 6.50 |
| 2 | 839 | 7.32 |
| 3 | 8.48 | 8.12 |
| 4 | 8.90 | 8.88 |
| 5 | 9.49 | 9.64 |
| 6 | 10.17 | 10.42 |
| 7 | 10.93 | 11.22 |
| 8 | 11.74 | 12.06 |
| 9 | 12.59 | 12.92 |
| 10 | 13.48 | 13.81 |

Estimating the parameters of specification [24] — again with the weighting mentioned in section 3.2.1 — resulted in the figures of Table 10.

Table 10

**Parameter results for equation [24]**

| Parameters | Variable $y_1$ | Variable $y_2$ |
|---|---|---|
| Condensed $\rho$ | 1.0414 | 1.0414 |
| Constant | -.0153 | .1815 |

Remarkable is the fact that the condensed $\rho$ is 25% higher than the original first order one, which is in line with the results shown by equation [24]; again Section 3.4 will continue the analysis.

### 3.3 Solutions

Various suggestions could be made neutralize time and space biases present in spatial econometric work.

One possibility is to test, if possible, given the data, several underlying specifications against each other, but the alternatives are rather numerous.

Another way of attacking the problem is to introduce time- and space flexible discount functions; some ideas about this follow.

The shape of a – spatial or temporal – lag function can be quite sophisticated; in fact, the **W** matrix of equation [12], normalized or not, can often be considered as an oversimplification of spatial interaction effects, and the same applies to time interaction. The reason is that distances, which are often implied, should not necessarily concern "physical" ones, but mostly refer to "functional" distances. e.g. economic structures (Kocornik-Mina, 2007)

Flexibility in this case can be introduced using some appropriate functional form, e.g. the well-known one-parameter Poisson (distribution) function $f(n) = e^{-\mu}\mu^n/n!,$ where $n$ is the time-lag degree; use of the latter in time series has shown that a first-order lag is not necessarily dominant (Agarwal, 2009).

The same applies to spatial lags; different possibilities exist for introducing flexibility, e.g. the use of a so-called Tanner function, specified as follows

$$f(d,\gamma)=\gamma^{*}de^{-\gamma d} \qquad [29]$$

with $d$ an appropriate distance function, $\gamma \geq 0$, and $\gamma^{*}$ the usual normalizing constant $\left(\gamma^{-2}\right)$.

Another possibility is the use of the following function

$$f(d,\gamma)=e^{(1-\gamma^{**})}\left[\ln(1+\gamma)+\gamma^{**}\right](1+\gamma d)^{-1} \qquad [30]$$

with again $\gamma \geq 0$. Both functions allow for a maximum effect, equation [30] showing it at a distance $d^{*}=\left(e^{(1-\gamma^{**})}-1\right)\big/\gamma$, so for $\gamma = 0$ the maximum effect operates at $d=\infty$, and for $\gamma=\infty$ at d=0 (further details in Ancot and Paelinck, 1983).

Data sometimes allow to compute intermingled space and time reactions (panel data, e.g.); how space and time have been combined will be illustrated by way of a bi-variate statistical approach. The specification problem can be approached as follows (Griffith and Paelinck, 2009).

As a distance function the degree of contiguity was selected, and years as time units. This implies that a discrete bi-variate function should be picked out, and a simple choice was the bi-variate Poisson function.

The non-normalized bi-variate Poisson distribution is specified as

$$f(n_1,n_2,r)=m_1^{\;n1}m_2^{\;n2}\left(n_1!n_2!\right)^{-1}\left[1+r(m_1m_2)^{1/2}p_{11}p_{21}+...\right], \qquad [31]$$

where $n_1$ and $n_2$ are the degrees of time- and space-lags, and where $\rho$ is the correlation coefficient between $n_1$ and $n_2$, $p_{11}$ and $p_{21}$ being the Charlier type $B$ polynomials (Ord, 1972) defined as:

$$p_k=\sum_{j=0}^{k}\binom{k}{j}m^{k-j}(-1)^{j}\,n_j, \qquad [32]$$

with $n_j$ denoting the frequency corresponding to index $j$. In the present case, $n_{1j}$ and $n_{2j}$ represent the relevant (space *and* time) lags.

The model was applied to eleven Belgian spatial units over the years 1995-2002; the following results show its applicability.

Parameters $\mu_1$ and $\mu_2$ were strictly positive, at the same time showing widely divergent values; $\rho$ was lying between -1 and +1, as it is required to do. Spatial parameters were all positive, again varying widely in value, and so were time parameters but with three exceptions; a mixed space-time parameter –together with $\rho$ typical for the space-time effect– was mostly negative, with four exceptions, the same applying for $\rho$, with three exceptions.

In terms of the lag coefficients, two spatial lags were peaking at contiguity 4, one at contiguity 3 and three at contiguity 2; for all other regions, the impact declined with increasing spatial lag, so altogether a striking variety of effects, due to the specific characteristics of the local economies considered. This is one of the main features of applied spatial econometrics; this point will be encountered frequently in spatial econometric work. In terms of the time lags, the farthest away peak was 7; another region peaked at 3, and four others at 2 all other regions showing a declining time effect. These latter results are classic, but the first ones mentioned are really astonishing.

As to $\rho$ of equation [31] and the mixed space-time parameter, this being the first occasion on which they seem to appear in combined space-time dynamics, little can at this time be said about their relative values, as more experience with them has to be gathered.

### 3.4 Applications

The discussion of Section 3.3 will now be applied to the simulations of Section 3.2.

#### *3.4.1 Models [21]-[22]*

A Poisson distribution specification will be used as a spatial discount function; remember its functional form $f(n)=e^{-\mu}\mu^n/n!$, where $n$ here denotes the degree of a time lag.

Using again the data of Table 7, and estimating by SDLS –with the weighting mentioned in section 3.2– resulted in $a$= .9524, $a^*$=1.2058 and $\mu$= .0268, results hardly different from those of Section 3.1, except for a slight increase in $a$. But the weakness of $\mu$ shows that there is very probably no *positive* second order effect, which invites to try a negative value for that effect, which would of course lead to the values of the parameters in equation [22].

#### *3.4.2 Models [23]-[24]*

The figures of Table 9 will again be used, again with the weighting of section 3.2. As there is only one "neighboring spatial unit" (the other series), no bi-variate Poisson distribution could be used, but two different distributions were used, one for the own time-lag, the other for the time lag of the other series. Table 11 presents the results of the SDLS estimation of the parameters, whose meanings are the following: $a$ (own reaction parameter), $b$ (pure contemporaneous spatial effect), $c$ (reaction parameter to the other series), $d$ (constant), $\mu$ (own Poisson lag parameter), $\mu^*$ (Poisson lag parameter of the other series); there is no correlation coefficient involved, as the two Poisson effects are not joint.

Table 11

**Parameters of re-estimating equations [28]**

| Parameters | Series 1 | Series 2 |
|---|---|---|
| $a$ | .3955 | .1049 |
| $b$ | .7803 | .7332 |
| $c$ | -.1632 | .1949 |
| $d$ | -.1117 | .4073 |
| $\mu$ | .0354 | .3931 |
| $\mu^*$ | .5646 | 1.7152 |

Considering the first series, coefficient $a$ is practically equal to that of equation [28a], .40, with a dominating first order effect ($\mu$<1). The coefficient for the other region ($c$) is negative, but this fact is compensated by the pure contemporaneous spatial effect ($b$), which has to be taken into account in the dynamics, due to the series' serial correlation; the Poisson parameter shows once more a dominating first order lag.

As to the second series, the effect of the other series is dominating, at the same time through $b$ and $c$, with a significant second order lag effect ($\mu^*$ >1). Without reproducing exactly equation [28b], the overall tendency of that equation is confirmed.

### 3.5 General approach

The problem could still be approached from a more general point of view.

Suppose there to be observations over $R$ regions and $T$ time periods; excluding own contemporary coefficients −for identification reasons− and also expectation-forward -effects, there are $C=R^2T$ estimable coefficients, including region-specific constants. As there are $O=RT$ observations, there remain $M=RT(R\text{-}1)$ missing observations.

Supposing all lagged observations to be available (these observations not appearing as endogenous variables; see the remark down Table 7, two possibilities remain. Either the parameters are considered to be non region-specific, in which case $R$–1=0, and all coefficients can be interpolated. Another possibility is to select $RT$ region-specific parameters, on the basis of an appropriate criterion; in the exercise to follow, the variation coefficient of the absolute parameter values, $\sigma/\mu$, was maximized, so as to obtain maximum contrasting impact effects (using real values might imply a $\mu$ near zero, with possible indeterminacy; this was the case with the figures of Table 11, leading up to $\mu$= .0231 and $\sigma/\mu$=56.8695).

These two possibilities were applied to the figures of Table 12.

Table 12

**Data for an application**

| T\R | 1 | 2 | 3 |
|---|---|---|---|
| -1 | 4 | 2 | 10 |
| 0 | 5 | 4 | 11 |
| 1 | 7 | 6 | 13 |
| 2 | 8 | 6 | 14 |
| 3 | `10 | 7 | 14 |

Only time periods 1, 2 and 3 were used in the calculations, periods 0 and -1 allowing only to estimate the lag parameters without discounting (see section 3.2); Table 13 presents the results.

Table 13

**Calculation results**

| Coeff.\Reg. | All R | $R_1$ | $R_2$ | $R_3$ |
|---|---|---|---|---|
| $a_1$ | -.3110 | | -.4091 | |
| $a_2$ | .8645 | | | 1.8333 |
| $a_3$ | -.3110 | -.1500 | | .6667 |
| $a_4$ | .8645 | .0300 | | -2.1667 |
| $a_5$ | .1652 | | | |
| $a_6$ | -.1561 | .0700 | | |
| $a_7$ | 17.1641 | | | |
| $a_8$ | -1 | | .0500 | |
| $a_9$ | -.5237 | | .3182 | |
| $\sigma/\mu$ | - | | .8632 | |

Column *R* refers to the first possibility (non region-specific parameters), columns $R_1$, $R_2$ and $R_3$ to the other case (as the parameters are region-specific, the 9 equations in the system are made up of three groups of three separate equations; appropriate binary variables assure the right split). $a_1$ and $a_2$ are the own first and second order lag coefficients; $a_3$ and $a_4$ apply to the same lags for the effects $R_1 \to R_2$, $R_2 \to R_1$, $R_1 \to R_3$, $a_5$ and $a_6$ to those for $R_3 \to R_1$, $R_3 \to R_2$, $R_2 \to R_3$; $a_7$ is the constant; $a_8$ is the spatial autoregression coefficient for $R_2 \to R_1$, $R_2 \to R_1$, $R_1 \to R_3$ and $a_9$ the one for $R_3 \to R_1$, $R_3 \to R_2$, $R_2 \to R_3$.

A typical equation is the following, *where x refers to* $R_1$, y to $R_2$, and z to $R_3$

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + a_3 y_{t-1} + a_4 y_{t-2} + a_5 z_{t-1} + a_6 z_{t-2} + a_7 + a_8 y_t + a_9 z_t \qquad [33]$$

As to the coefficients, one notices large differences according to the specifications, and also the absence of a constant in the region-specific case; this is again an example of multiple regimes ruling interregional behavior (Griffith and Paelinck, 2011, Part 2, Chapter 13). Of note also is the fact that the region-specific parameters should include the effects of the relevant topology.

## 4. General conclusions

From the spatial econometric exercises reported in this study, a certain number of conclusions have been drawn which will be reported here again.

It has been shown first that a common rule of certain econometric models –to wit input-output ones– can be an important help, together with topological information, in identifying and re-estimating a multi-regional model. More generally it has been concluded that a study of the family links between spatial econometric specifications is a valuable topic to be envisaged for further study, and that an intrusion into general system theory could also be advocated.

Concerning the use of flexible spatio-temporal weight functions (see the earlier mentioned Ancot-Paelinck, 1983, and Griffith-Paelinck, 2011, studies), only a few tests are available but they tend to show that the choice of such functions can correctly specify complex space-time reaction patterns. An obvious extension would be the choice of more complex functional expressions (bi-variate polynomials, e.g.), a topic for further investigations. As to the region-specific case treated in section 3.5, the availability of large *T* multi-regional series should allow to increase the number of region-specific parameters, limiting e.g. the lags to only a few orders.

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