The gamma flexible Weibull distribution: Properties and Applications

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Abstract: A new gamma flexible Weibull distribution is introduced, which presents a bathtub-shaped hazard rate, and some of its properties are obtained. The parameters are estimated via maximum likelihood, and a simulation study is performed to examine the consistency of the estimates. The utility of the proposed model is shown using three real applications.

Keywords: Bathtub, Bimodal, COVID-19, Maximum likelihood, Moment, Quantile function

MSC: 33B05, 60E05, 62P99, 65C05

1 Introduction

Various phenomena that occur in the real world can be explained by statistical distributions. For a long time, many of the common distributions (Weibull, gamma, Burr XII, Gumbel) were sufficient for this purpose. However, with computer science development, more flexible distributions have become mandatory. One way to generate new families of distribution is through techniques to generalize existing ones. The main characteristic of these generalizations is the addition of more parameters to their baseline distributions, thus increasing their flexibility.

The Weibull distribution is widely used in many fields, but it is not suitable for bathtub-shaped or unimodal hazard rates. Thus, several models have been developed to extend this distribution and increase the modeling ability, such as those in (Mudholkar and Srivastava, 1993), (Xie and Lai, 1996), (Xie et al., 2002), (Lai et al., 2003), (Famoye et al., 2005), and (Cordeiro et al., 2010), among others.

Of the various modifications made to the Weibull distribution, the one of interest in this article is the flexible Weibull (FW) distribution (Bebbington et al., 2007) with shape parameters $\alpha, \beta > 0$, cumulative distribution function (cdf)

$$G(x; \alpha, \beta) = 1 - \exp\left(-e^{\alpha x - \frac{\beta}{\beta}}\right), \quad x > 0,$$
and probability density function (pdf)

\[ g(x; \alpha, \beta) = \left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} \exp \left( -e^{\alpha x - \frac{\beta}{x}} \right). \]

For \( \beta = 0 \) and \( \alpha = \log(\lambda) \), the FW model reduces to the exponential, and then it can be regarded as a generalization of the Weibull (Bebbington et al., 2007).

There are several extensions of the FW distribution such as those reported by (El-Gohary et al., 2015), (El-Desouky et al., 2016), (Mustafa et al., 2016), (El-Damcese et al., 2016), (El-Desouky et al., 2017), and (Ahmad and Iqbal, 2017).

Zografos and Balakrishnan (2009) and Risti´c and Balakrishnan (2012) defined the cdf of the gamma-G class for any parent cdf \( G(x) = G(x; \theta) \) with parameter vector \( \theta \) of dimension \( p \), by (for \( x \in \mathbb{R} \))

\[ F(x) = F(x; a, \theta) = \frac{\gamma(a, -\log(1 - G(x)))}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{-\log(1-G(x))} t^{a-1} e^{-t} dt, \quad (1) \]

where \( a > 0 \) is a shape parameter, and \( \Gamma(\cdot) \) is the gamma function. For \( a = 1 \), Equation (1) reduces to the parent G cdf.

Recently, the gamma-G family has received considerable attention in works by (Nadarajah et al., 2015), (Alzaatreh et al., 2014), (Nadarajah et al., 2015), (Cordeiro et al., 2016), (Bourguignon and Cordeiro, 2016), (Iriarte et al., 2017), (Guerra et al., 2017), and (David et al., 2021), among others.

The article unfolds as follows: Section 2 defines the gamma-flexible Weibull (GFW) distribution and a linear representation for its density. The moments and generating function are reported in Section 3. Section 4 estimates the parameters by the maximum likelihood method and conducts a simulation study. Three real data sets are analyzed in Section 5 to show the utility of the new model. Finally, we draw some conclusions in Section 6.

## 2 The GFW model and its linear representation

A random variable \( X \) follows the GFW distribution, say \( X \sim \text{GFW}(a, \alpha, \beta) \), if its cdf and pdf (omitting parameters in the functions) are

\[ F(x) = F(x; a, \theta) = \frac{\gamma(a, -\log(1 - G(x)))}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{-\log(1-G(x))} t^{a-1} e^{-t} dt, \quad (2) \]

and

\[ f(x) = \frac{\left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} \exp \left( -e^{\alpha x - \frac{\beta}{x}} \right)}{\Gamma(a)}, \quad (3) \]

respectively.

The FW distribution was introduced in engineering, but it can be used in several fields. So, the GFW distribution can also be adopted in a similar manner.

The hazard rate function (hrf) of \( X \) follows from the last two expressions.

The GFW is identical to the FW distribution when \( a = 1 \). The calculations in all sections were done using R software (R Core Team, 2020).

Figure 1 displays some plots of the density of \( X \), which can be symmetric, right-symmetric, left-symmetric, or bimodal. Plots of the hrf of \( X \) are reported in Figure 2, which has increasing, decreasing, bathtub, and unimodal shapes.
A simple motivation for the GFW distribution follows from Zografos and Balakrishnan (2009), where the GFW density can be approximated by the upper record value density from a sequence of independent and identically distributed FW random variables. Further, we highlight the utility of the proposed distribution in medical data analysis. In fact, the GFW distribution can be selected
as the best model, especially in modeling unimodal and bimodal data of COVID-19 and cancer as illustrated in Section 5.

Following the concept of exponentiated distributions (Cordeiro et al., 2013), the exponentiated FW ("expFW") cdf with power parameter $\delta$, say $\text{EFW}(\alpha, \beta, \delta)$ (for $x > 0$), is

$$H_\delta(x; \alpha, \beta) = \left[1 - \exp\left(-e^{\alpha x - \frac{\delta}{x}}\right)\right]^{\delta}$$

and the corresponding pdf reduces to

$$h_\delta(x; \alpha, \beta) = \delta \left(\alpha + \frac{\beta}{x}\right) e^{\alpha x - \frac{\beta}{x}} \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right) \left[1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right]^{\delta-1}.$$  

From Proposition 2 of Castellares and Lemonte (2015), we can write

$$[-\ln(1-v)]^c = v^c \sum_{m=0}^{\infty} \rho_m(c) v^m,$$  

where $c \in \mathbb{R}$, $|v| < 1$, $\rho_0(c) = 1$, $\rho_m(c) = c \psi_{m-1}(m + c - 1)$ for $m \geq 1$, and $\psi_m(\cdot)$ are Stirling polynomials, namely

$$\psi_{m-1}(w) = \frac{(-1)^n}{(n+1)!} \left[ T_{n+1} - \frac{w + 2}{n + 2} T_{n+2} + \frac{(w+2)(w+3)}{(n+2)(n+3)} T_{n+3} - \cdots \right. 
$$

$$+ \left. (-1)^n \frac{(w+2)(w+3) \cdots (w+n)}{n+2)(n+3) \cdots (2n)} T_n \right],$$  

where $T_n = (2n+1-m) T_{n+1} + (n-m+1) T_{n+1}^{-1}$ are positive integers, $T_0 = 1$, $T_{n+1} = 1 \times 3 \times 5 \times \cdots \times (2n+1)$, and $T_{n+1}^m = 1$.

From Equation (4), we can rewrite Equation (3) as (Castellares and Lemonte, 2015)

$$f(x; a, \alpha, \beta) = \sum_{m=0}^{\infty} p_m h_{m+a}(x; \alpha, \beta),$$  

where $\varphi_0(a) = \Gamma(a)^{-1}$, $p_m = p_m(a) = \varphi_m(a)/(m + a)$, $\varphi_m(a) = \Gamma(a)^{-1} \rho_m(a - 1) = (a - 1) \Gamma(a)^{-1} \psi_{m-1}(m + a - 2)$ (for $m \geq 1$) can be determined from (5), and $h_{m+a}(x; \alpha, \beta)$ denotes the EFW density with power parameter $m + a$.

Equation (6) reveals that the GFW density is a linear combination of EFW densities. So, its properties can follow from those of the EFW distribution.

### 3 Moments and generating function

We calculate numerically in Table 1 the first four moments, standard deviation (SD), skewness (SK) and kurtosis (KR) of $X$ varying $a$ and $\beta$, with $\alpha = 0.04$. The moments increase and the skewness and kurtosis decrease if $\beta$ increases for $a$ fixed. Note that the same happen when $a$ increases for $\beta$ fixed. 

If $Y_{m+a} \sim \text{EFW}(m + a, \alpha, \beta)$, we write from Equation (6)

$$\mu_r = \mathbb{E}(X^r) = \sum_{m=0}^{\infty} p_m \mathbb{E} \left(Y_{m+a}^r\right).$$  

(7)
Further, the \( r \)th moment of the EFW distribution is

\[
\mathbb{E}(Y_m^r) = (m + a) \int_0^\infty x^r \left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} \exp \left[ -e^{\alpha x - \frac{\beta}{x}} \right] \left[ 1 - \exp \left( -e^{\alpha x - \frac{\beta}{x}} \right) \right]^{m+a-1},
\]

where \( \left[ 1 - \exp \left( -e^{\alpha x - \frac{\beta}{x}} \right) \right]^{m+a-1} \) can be written as

\[
\left[ 1 - \exp \left( -e^{\alpha x - \frac{\beta}{x}} \right) \right]^{m+a-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+a)}{j! \Gamma(m+a-j)} \exp \left( -je^{\alpha x - \frac{\beta}{x}} \right),
\]

and then

\[
\mathbb{E}(Y_m^r) = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+a+1)}{j! \Gamma(m+a-j)} \int_0^\infty x^r \left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} \left( 1 + \frac{\beta}{x} e^{\alpha x - \frac{\beta}{x}} \right) \left[ 1 - (j+1)e^{\alpha x - \frac{\beta}{x}} \right] \, dx.
\]
By using power series for \( \exp \left[ -(j+1)e^{\alpha x - \frac{\beta}{x}} \right] \) and \( e^{2(k+1)\alpha x} \) gives

\[
\mathbb{E}(Y_{m+a}^r) = \sum_{j,k,i=0}^{\infty} \frac{(-1)^{j+k} (j+1)^k 2^i (k+1)^i \Gamma(m+a+1)\alpha^i}{j!k!i!\Gamma(m+a-j)} \times \int_0^{\infty} x^{r+i} \left( \alpha + \frac{\beta}{x^2} \right) e^{-(k+1)\alpha x - \frac{(k+1)\beta}{x}} dx.
\]  

(8)

Based on the result (3.471 9) in Gradshteyn and Ryzhik (2007), we obtain

\[
\mathbb{E}(Y_{m+a}^r) = \sum_{j,k,i=0}^{\infty} \frac{(-1)^{j+k} (j+1)^k 2^i (k+1)^i \Gamma(m+a+1)\alpha^i}{j!k!i!\Gamma(m+a-j)} \times \left[ 2\alpha \left( \frac{\beta}{\alpha} \right)^{\nu+1} K_{\nu+1} \left( 2(k+1)\sqrt{\alpha\beta} \right) + 2\beta \left( \frac{\beta}{\alpha} \right)^{\nu+1} K_{\nu-1} \left( 2(k+1)\sqrt{\alpha\beta} \right) \right],
\]

(9)

where

\[
\nu = r+i, \quad K_{\nu}(z) = \frac{\pi \csc(\pi \nu)}{2} [I_{-\nu}(z) - I_{\nu}(z)], \quad \text{and} \quad I_{\nu}(z) = \sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\ell+\nu+1)\ell!} \left( \frac{z}{2} \right)^{2\ell+\nu}
\]

are the modified Bessel functions of the second and first kind, respectively (for \( \nu \notin \mathbb{Z} \)).

Substituting (9) into (7) gives the \( r \)th moment of the GFW distribution.

In a similar manner, the \( r \)th incomplete moment of \( X \), say \( m_r(s) = \int_0^s x^r f(x) \, dx \), follows as

\[
m_r(s) = \sum_{j,k,i=0}^{\infty} \frac{(-1)^{j+k} (j+1)^k 2^i (k+1)^i p_m \Gamma(m+a+1)\alpha^i}{j!k!i!\Gamma(m+a-j)} \times \int_0^{s} x^{r+i} \left( \alpha + \frac{\beta}{x^2} \right) e^{-(k+1)\alpha x - \frac{(k+1)\beta}{x}} dx.
\]

From Theorem 2 of Chaudhry and Zubair (1994), we obtain (for \( r \geq 1 \))

\[
m_r(s) = \sum_{m,j,k,i=0}^{\infty} \frac{(-1)^{j+k} (j+1)^k 2^i p_m \Gamma(m+a+1)}{j!k!i!(k+1)^r \Gamma(m+a-j) \alpha^r} \times \left\{ \gamma \left( (k+1)\alpha s; (r+1), (k+1)^2\alpha\beta \right) \right\}_0^{(k+1)} + \left( (k+1)\gamma \left( (k+1)\alpha s; (r+i-1), (k+1)^2\alpha\beta \right) \alpha\beta \right\},
\]

where \( \gamma(x; a, b) = \int_0^x t^{a-1} e^{-t/b} dt \) is the generalized lower incomplete gamma function.

The generating function (gf) of \( X \) can be written from (6) as

\[
M(t) = \sum_{m=0}^{\infty} p_m M_{m+a}(t),
\]

(10)
where \( M_{m+a}(t) \) is the gf of \( Y_{m+a} \). The gf of the EFW distribution is

\[
M_{m+a}(t) = (m + a) \int_0^\infty e^{tx} \left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} \exp \left( -e^{\alpha x - \frac{\beta}{x}} \right) \left[ 1 - \exp \left( -e^{\alpha x - \frac{\beta}{x}} \right) \right]^{m+a-1}.
\]

Following a similar algebra as for Equation (8) and again the result (3.471.9) (Gradshteyn and Ryzhik, 2007), we obtain (for \( t < \alpha \))

\[
M_{m+a}(t) = \sum_{j,k,i=0}^\infty \frac{(-1)^{j+k}(j+1)^k 2^i (k+1)^i \alpha^i \Gamma(m+a+1)}{j!k!i!\Gamma(m+a-j)}
\]

\[
\times \left\{ 2\alpha \left[ \frac{(k+1)\beta}{(k+1)\alpha - t} \right]^{i+1} K_{i+1} \left( 2\sqrt{[(k+1)\alpha - t]/(k+1)\beta} \right) 
\right.
\]

\[
\left. + 2\beta \left[ \frac{(k+1)\beta}{(k+1)\alpha - t} \right]^{i+1} K_{i-1} \left( 2\sqrt{[(k+1)\alpha - t]/(k+1)\beta} \right) \right\}.
\]

Substituting Equation (11) into (10) gives the gf of the GFW distribution.

The quantile function (qf) of the FW distribution is given by (Bebbington et al., 2007)

\[
Q_{FW}(u; \alpha, \beta) = \frac{1}{2\alpha} \left\{ \log \left[ -\log (1-u) \right] + \sqrt{\left\{ \log \left[ -\log (1-u) \right] \right\}^2 + 4\alpha\beta} \right\}.
\]

By inverting (2) and using results in Nadarajah et al. (2015), the qf of \( X \) follows as (for \( 0 < u < 1 \))

\[
Q_{GFW}(u; a, \alpha, \beta) = \frac{1}{2\alpha} \left\{ \log \left[ Q^{-1}[a, (1-u)] \right] + \sqrt{\left\{ \log \left[ Q^{-1}[a, (1-u)] \right] \right\}^2 + 4\alpha\beta} \right\},
\]

where \( Q^{-1}(a, u) \) is the inverse function of \( Q(a, x) = 1 - \gamma(a, x)/\Gamma(a) \).

Approximations for the skewness and kurtosis of \( X \) can be based on quantile measures from (12). Let \( Q_{GFW}(u) = Q_{GFW}(u; a, \alpha, \beta) \). The Bowley’s skewness (Kenney and Keeping, 1962) is

\[
B(a, \alpha, \beta) = \frac{Q_{GFW}(3/4) + Q_{GFW}(1/4) - 2Q_{GFW}(1/2)}{Q_{GFW}(3/4) - Q_{GFW}(1/4)},
\]

whereas the Moors kurtosis (Moors, 1988) is

\[
M(a, \alpha, \beta) = \frac{Q_{GFW}(7/8) - Q_{GFW}(5/8) - Q_{GFW}(3/8) + Q_{GFW}(1/8)}{Q_{GFW}(6/8) - Q_{GFW}(2/8)}.
\]

Plots of these quantities for some choices of \( \alpha \) and \( \beta \) as functions of \( a \) are reported in Figure 3. Note that the skewness increases when \( a \) goes to one and decreases from this value. The kurtosis decreases rapidly for small values of \( a \) and stabilizes when \( a \) increases.

An application of (12) using the first incomplete moment \( m_1(s) \) refers to the Bonferroni and Lorenz curves defined by (for a given probability \( \pi \))

\[
B(\pi) = \frac{m_1(q)}{\pi \mu_1} \quad \text{and} \quad L(\pi) = \frac{m_1(q)}{\mu_1},
\]

respectively, where \( q = Q_{GFW}(\pi) \). Plots of these curves versus \( \pi \) for some choices of \( a \) (with \( \alpha = 0.01 \) and \( \beta = 15 \)) are displayed in Figure 4.
4 Estimation and Simulations

The log-likelihood function for $\theta = (a, \alpha, \beta)^\top$ given the data set $x_1, \ldots, x_n$ from $X$ is

$$
\ell(\theta) = \sum_{i=1}^{n} \log \left( \alpha + \frac{\beta}{x_i^a} \right) + a \sum_{i=1}^{n} \left( \alpha x_i - \frac{\beta}{x_i^a} \right) + \sum_{i=1}^{n} \left( -e^{\alpha x_i - \frac{\beta}{x_i^a}} \right) - n \log[\Gamma(a)].
$$

(13)
The maximum likelihood estimate (MLE) of $\theta$, say $\hat{\theta}$, can be found by maximizing Equation (13) numerically using scripts such as optim or nlm in R, MaxBFGS in Ox, and PROC NLMIXED in SAS.

We generate 1,000 Monte Carlo replicates for the GFW model from Equation (12) with sample sizes $n = 50, 100, 300,$ and $500$ under three scenarios: $(a, \alpha, \beta) = (0.9, 2, 1.2)$ for scenario I; $(a, \alpha, \beta) = (0.5, 1.5, 3)$ for scenario II; and $(a, \alpha, \beta) = (1.5, 0.5, 0.8)$ for scenario III. We use the optim script of R to maximize (13). The averages, biases and mean square errors (MSEs) of the estimates are listed in Table 2. The averages tend to the true parameter values and the biases and MSEs converge to zero when $n$ increases, which reveal that the MLEs are consistent.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Parameter</th>
<th>Average</th>
<th>Bias</th>
<th>MSE</th>
<th>Average</th>
<th>Bias</th>
<th>MSE</th>
<th>Average</th>
<th>Bias</th>
<th>MSE</th>
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<td>50</td>
<td>$a$</td>
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<td>0.261</td>
<td>0.630</td>
<td>0.130</td>
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<td>1.480</td>
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<td>0.815</td>
<td>4.787</td>
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<td>2.782</td>
<td>0.514</td>
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<tr>
<td></td>
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<td>1.112</td>
<td>8.537</td>
<td>4.623</td>
<td>1.623</td>
<td>26.617</td>
<td>1.044</td>
<td>0.244</td>
<td>0.574</td>
</tr>
<tr>
<td>100</td>
<td>$a$</td>
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<td>-0.090</td>
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<td>0.588</td>
<td>0.088</td>
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<td>0.430</td>
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<td>0.836</td>
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<td>0.009</td>
<td>0.002</td>
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<td>$a$</td>
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<td>0.156</td>
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<td>0.083</td>
<td>0.784</td>
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Table 2: Simulation results for the GFW model.

5 Applications

We present three applications of the new model and compare it to other distributions: exponentiated Weibull (EW) (Mudholkar and Srivastava, 1993), modified Weibull (MW) (Lai et al., 2003), beta Weibull (BW) (Famoye et al., 2005), FW, Kumaraswamy Weibull (KwW) (Cordeiro et al., 2010), and Kumaraswamy Burr XII (KwBXII) (Paranaiba et al., 2013).

The best model is chosen based on Cramér-von Mises ($W^*$), Anderson-Darling ($A^*$), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). The MLEs, standard errors (SEs), and the statistics are found using the AdequacyModel script (Marinho et al., 2019) of R software.

5.1 Failure times

The failure times of 50 components (per 1000h) are (Murthy et al., 2004): 0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.041, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080.
Table 3 gives some descriptive statistics. The mean is greater than the median, and then the data are right-skewed and leptokurtic.

<table>
<thead>
<tr>
<th>Mean</th>
<th>Median</th>
<th>SD</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min.</th>
<th>Max.</th>
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</tr>
</tbody>
</table>

Table 3: Descriptive statistics for failure times.

Table 4 reports the MLEs and their SEs (in parentheses). The MW, EW, KwW, and BW models have higher SEs related to their estimates, whereas the GFW, FW, and KwBXII models have accurate estimates.

Table 5 indicates that the GFW model gives the best fit to the data since it has the lowest statistics among all models. The generalized likelihood ratio (GLR) test (Vuong, 1989) is used to compare the GFW model against the FW ($GLR = 3.911$), MW ($GLR = 3.503$), EW ($GLR = 3.455$), KwW ($GLR = 3.372$), KwBXII ($GLR = 3.452$), and BW ($GLR = 3.450$) models for a significance level of 5%. These results show that the GFW distribution provides the best fit to the current data.

The plots of the estimated densities and estimated survival functions for the most competitive models are shown in Figure 5. The GFW distribution provides the closest approximations to the histogram and empirical survival function, which shows its utility for real-life applications.

<table>
<thead>
<tr>
<th>Model</th>
<th>MLEs (SEs)</th>
<th>MLEs (SEs)</th>
<th>MLEs (SEs)</th>
<th>MLEs (SEs)</th>
<th>MLEs (SEs)</th>
<th>MLEs (SEs)</th>
<th>MLEs (SEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GFW ($\alpha, \alpha, \beta$)</td>
<td>1.362 (0.190)</td>
<td>0.109 (0.013)</td>
<td>0.126 (0.033)</td>
<td>FW ($\alpha, \beta$)</td>
<td>0.099 (0.012)</td>
<td>0.183 (0.034)</td>
<td>0.13 (0.03)</td>
</tr>
<tr>
<td>MW ($\alpha, \lambda, \beta$)</td>
<td>0.496 (0.099)</td>
<td>0.034 (0.025)</td>
<td>0.562 (0.098)</td>
<td>EW ($\alpha, \lambda, \beta$)</td>
<td>0.290 (0.681)</td>
<td>0.770 (0.990)</td>
<td>0.785 (1.546)</td>
</tr>
<tr>
<td>KwW ($\alpha, b, \alpha, \beta$)</td>
<td>0.118 (0.024)</td>
<td>2.368 (1.555)</td>
<td>4.551 (0.099)</td>
<td>KwBXII ($\alpha, b, c, k, s$)</td>
<td>0.121 (0.019)</td>
<td>2.199 (0.477)</td>
<td>4.381 (0.147)</td>
</tr>
<tr>
<td>BW ($\alpha, b, \alpha, \beta$)</td>
<td>0.708 (1.392)</td>
<td>0.703 (1.460)</td>
<td>0.412 (1.575)</td>
<td>0.819 (1.057)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Findings from the fitted models to failure times.

5.2 COVID-19

Table 5: Adequacy measures for the models fitted to failure times.

<table>
<thead>
<tr>
<th>Model</th>
<th>$W^*$</th>
<th>$A^*$</th>
<th>AIC</th>
<th>CAIC</th>
<th>BIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>GFW</td>
<td>0.042</td>
<td>0.257</td>
<td>193.850</td>
<td>194.372</td>
<td>199.586</td>
<td>196.035</td>
</tr>
<tr>
<td>FW</td>
<td>0.079</td>
<td>0.414</td>
<td>195.846</td>
<td>196.101</td>
<td>199.670</td>
<td>197.302</td>
</tr>
<tr>
<td>MW</td>
<td>0.130</td>
<td>0.850</td>
<td>208.727</td>
<td>209.249</td>
<td>214.463</td>
<td>210.912</td>
</tr>
<tr>
<td>EW</td>
<td>0.150</td>
<td>0.946</td>
<td>210.713</td>
<td>211.234</td>
<td>216.449</td>
<td>212.897</td>
</tr>
<tr>
<td>KwW</td>
<td>0.131</td>
<td>0.861</td>
<td>210.706</td>
<td>211.595</td>
<td>218.355</td>
<td>213.619</td>
</tr>
<tr>
<td>BW</td>
<td>0.149</td>
<td>0.942</td>
<td>212.696</td>
<td>213.585</td>
<td>220.344</td>
<td>215.608</td>
</tr>
<tr>
<td>KwBXII</td>
<td>1.132</td>
<td>0.870</td>
<td>213.086</td>
<td>214.450</td>
<td>222.646</td>
<td>216.726</td>
</tr>
</tbody>
</table>

Table 6 shows some descriptive statistics for these data. The skewness is positive, and the kurtosis indicates mesokurtic distribution. The MLEs and their SEs (in parentheses) reported in Table 7 reveal that the GFW, FW, and KwW distributions have accurate estimates, and the other ones have high SEs relative to their estimates. The results in Table 8 indicate that the GFW model has the lowest values of the criteria, so it can be chosen as the best model. Additionally, the GLR test also reveals that the GFW model is better than the FW ($GLR = 3.383$), MW ($GLR = 3.961$), EW ($GLR = 2.925$), KwW

Table 6: Descriptive statistics for COVID-19 data.

<table>
<thead>
<tr>
<th>Mean</th>
<th>Median</th>
<th>SD</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>67.8670</td>
<td>57.0000</td>
<td>48.3850</td>
<td>2341.1</td>
<td>0.8198</td>
<td>2.9783</td>
<td>4</td>
<td>201</td>
</tr>
</tbody>
</table>
(GLR = 2.473), KwBXII (GLR = 3.502), and BW (GLR = 4.698) models for a significance level of 5%.

Figure 6 reports plots of the estimated densities and estimated cumulative functions for the most adequate models. The fit of the new distribution is closer to the histogram and empirical cumulative function than those of the other distributions. So, these results support that the GFW distribution is better suited to the current data.

Table 7: Findings from the fitted models to COVID-19 data.

<table>
<thead>
<tr>
<th>Model</th>
<th>MLEs (SEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GFW (a, α, β)</td>
<td>1.702 (0.253)</td>
</tr>
<tr>
<td>FW (α, β)</td>
<td>0.008 (0.001)</td>
</tr>
<tr>
<td>MW (α, β, λ)</td>
<td>0.005 (0.002)</td>
</tr>
<tr>
<td>EW (α, β, λ)</td>
<td>0.013 (0.005)</td>
</tr>
<tr>
<td>KwW (a, b, α, β)</td>
<td>1.333 (0.083)</td>
</tr>
<tr>
<td>KwBXII (a, b, c, k, s)</td>
<td>10.526 (25.224)</td>
</tr>
<tr>
<td>BW (a, b, α, β)</td>
<td>3.697 (1.303)</td>
</tr>
</tbody>
</table>

Table 8: Adequacy measures for the models fitted to COVID-19 data.

<table>
<thead>
<tr>
<th>Model</th>
<th>W*</th>
<th>A*</th>
<th>AIC</th>
<th>CAIC</th>
<th>BIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>GFW</td>
<td>0.034</td>
<td>0.241</td>
<td>855.333</td>
<td>855.637</td>
<td>862.590</td>
<td>858.248</td>
</tr>
<tr>
<td>FW</td>
<td>0.095</td>
<td>0.596</td>
<td>859.516</td>
<td>859.666</td>
<td>864.353</td>
<td>861.459</td>
</tr>
<tr>
<td>MW</td>
<td>0.057</td>
<td>0.348</td>
<td>858.767</td>
<td>859.071</td>
<td>866.024</td>
<td>861.682</td>
</tr>
<tr>
<td>EW</td>
<td>0.059</td>
<td>0.351</td>
<td>858.690</td>
<td>858.994</td>
<td>865.946</td>
<td>861.605</td>
</tr>
<tr>
<td>KwW</td>
<td>0.056</td>
<td>0.335</td>
<td>860.250</td>
<td>860.763</td>
<td>869.925</td>
<td>864.137</td>
</tr>
<tr>
<td>BW</td>
<td>0.088</td>
<td>0.544</td>
<td>863.876</td>
<td>864.389</td>
<td>873.551</td>
<td>867.763</td>
</tr>
<tr>
<td>KwBXII</td>
<td>0.083</td>
<td>0.508</td>
<td>864.870</td>
<td>865.649</td>
<td>876.964</td>
<td>869.728</td>
</tr>
</tbody>
</table>

5.3 Laryngeal cancer

The data set corresponds to the lifetime (in months) of 90 male patients with laryngeal cancer. The data are (Colosimo and Giolo, 2006): 0.6, 1.3, 2.4, 3.2, 3.3, 3.5, 3.5, 3.6, 4.0, 4.0, 4.3, 5.3, 6.0, 6.4, 6.5, 7.4, 2.5, 3.2, 3.3, 4.5, 4.5, 5.5, 5.9, 5.9, 6.1, 6.2, 6.5, 6.7, 7.0, 7.4, 8.1, 8.1, 9.6, 10.7, 0.2, 1.8, 2.0, 3.6, 4.0, 6.2, 7.0, 2.2, 2.6, 3.3, 3.6, 4.3, 4.3, 5.0, 7.5, 7.6, 9.3, 0.3, 0.3, 0.5, 0.7, 0.8, 1.0, 1.3, 1.6, 1.8, 1.9, 1.9, 3.2, 3.5, 5.0, 6.3, 6.4, 7.8, 3.7, 4.5, 4.8, 4.8, 5.0, 5.1, 6.5, 8.0, 9.3, 10.1, 0.1, 0.3, 0.4, 0.8, 0.8, 1.0, 1.5, 2.0, 2.3, 3.6, 3.8, 2.9, 4.3.

Some descriptive statistics in Table 9 reveal that the data are right-skewed and platykurtic. For these data, we compare the GFW distribution with other models that also have the bimodal shape,
Figure 6: (a) Estimated densities of three models; (b) empirical and estimated cumulative functions of the models.

namely, the Odd log-logistic flexible Weibull (OLLFW) (Prataviera et al., 2018), extended Weibull log-logistic (EWLL) (Abouelmagd et al., 2019), Marshall-Olkin flexible Weibull (MOFW) (Mustafa et al., 2016), and FW.

<table>
<thead>
<tr>
<th>Mean</th>
<th>Median</th>
<th>SD</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.197</td>
<td>4</td>
<td>2.612</td>
<td>6.901</td>
<td>0.343</td>
<td>2.367</td>
<td>0.1</td>
<td>10.700</td>
</tr>
</tbody>
</table>

Table 9: Descriptive statistics for laryngeal cancer data.

The MLEs and their corresponding SEs (in parentheses) in Table (10) show that the GFW, OLLFW, MOFW, and FW distributions have accurate estimates. The GFW distribution has the lowest values of the adequacy measures in Table (11) and can provide a better fit than the other distributions. The GLR test confirms that the GFW distribution fits the current data better than the OLLFW (GLR = 4.657), EWLL (GLR = 3.741), MOFW (GLR = 11.556), and FW (GLR = 3.603) distributions for a significance level of 5%. The plots in Figure 7 also support our claim.

6 Conclusions

We introduced a new versatile distribution called the gamma flexible Weibull and provided some of its properties. A simulation study demonstrated that the maximum likelihood estimates of the parameters are consistent. Three real applications showed that the new distribution is extremely competitive to other lifetime models for unimodal and bimodal medical data.
Table 10: Findings from the fitted models to laryngeal cancer data.

<table>
<thead>
<tr>
<th>Model</th>
<th>MLEs (SEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GFW ((a, \alpha, \beta))</td>
<td>2.527 (0.223), 0.201 (0.012), 0.197 (0.070)</td>
</tr>
<tr>
<td>OLLFW ((a, \alpha, \beta))</td>
<td>0.359 (0.053), 0.288 (0.023), 2.869 (0.106)</td>
</tr>
<tr>
<td>EWLL (\lambda, \alpha, \beta)</td>
<td>0.072 (0.459), 0.925 (0.457), 21.689 (3.756)</td>
</tr>
<tr>
<td>MOFW ((a, \alpha, \beta))</td>
<td>6.118 (1.528), 0.188 (0.011), 0.453 (0.106)</td>
</tr>
<tr>
<td>FW (\alpha, \beta)</td>
<td>0.142 (0.012), 1.286 (0.198)</td>
</tr>
</tbody>
</table>

Table 11: Adequacy measures for the models fitted to laryngeal cancer data.

<table>
<thead>
<tr>
<th>Model</th>
<th>(W^*)</th>
<th>(A^*)</th>
<th>AIC</th>
<th>CAIC</th>
<th>BIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>GFW</td>
<td>0.035</td>
<td>0.211</td>
<td>414.251</td>
<td>414.530</td>
<td>421.751</td>
<td>417.275</td>
</tr>
<tr>
<td>OLLFW</td>
<td>0.310</td>
<td>1.744</td>
<td>440.411</td>
<td>440.690</td>
<td>447.910</td>
<td>443.435</td>
</tr>
<tr>
<td>EWLL</td>
<td>0.148</td>
<td>0.913</td>
<td>425.014</td>
<td>425.293</td>
<td>432.514</td>
<td>428.038</td>
</tr>
<tr>
<td>MOFW</td>
<td>0.048</td>
<td>0.278</td>
<td>415.136</td>
<td>415.415</td>
<td>422.635</td>
<td>418.160</td>
</tr>
<tr>
<td>FW</td>
<td>0.564</td>
<td>3.218</td>
<td>462.138</td>
<td>462.275</td>
<td>467.137</td>
<td>464.154</td>
</tr>
</tbody>
</table>

Figure 7: (a) Estimated densities of three models; (b) empirical and estimated survival functions of the models.
Acknowledgments

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References


