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REGULAR ARTICLE

Semiparametric von Mises kernel circular density estimator

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Abstract: In this paper, we propose to estimate the circular density function by the semiparametric bias-corrected circular kernel method using the particular von Mises kernel. This method consists to apply a multiplicative bias correction for the initial parametric model in order to improve the quality of the estimator as well as the bias. Two semiparametric estimators Hjort and Glad (1995) (HG) and Jones, Signorini, and Hjort (1999) (JSH) for probability density estimation are applied on circular data with support $[0, 2\pi)$. The properties of the latter are reported such as the bias, the variance and the mean square error integrated (MISE). A comparative study is performed to evaluate the performance of the semiparametric estimator (HG and JSH). The popular cross validation technique is adapted for bandwidth selection. A simulation and a real data application for circular data illustrate in terms of integrated squared bias (ISB) and integrated squared error (ISE) that the semiparametric estimators JSH and JLN with the von Mises kernel perform better than the classical and HG estimators.

Keywords: Bandwidth selection, Circular data, Cross validation, Multiplicative bias correction MBC, von Mises kernel

MSC: 60E05, 62P99

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1 Introduction

In recent years, circular data analysis has received a lot of interest in the statistical literature, because they appear in several areas, such as biology (Batschelet, 1981), ecology (Jammalamadaka and Lund, 2006), meteorology (Bowers et al., 2000), sociology (Brunsdon and Corcoran, 2006), medicine (Mooney et al., 2003) and biomechanics (Mann et al., 2003). Given a random sample $\Theta_1, \Theta_2, \ldots, \Theta_n$

which take values in $[0, 2\pi)$ from some unknown density f, the nonparametric kernel density estimator of circular data proposed by (Hall et al., 1987), is given by $\hat{f}(\theta) = (1/n)\sum_{i=1}^{n} k_\nu(\theta - \Theta_i)$, where $\theta > 0$ is the estimation point and ν is the bandwidth parameter, this estimator is applied by several authors: (Bai et al., 1988), (Klemelä, 2000), (Taylor, 2008), (Marzio et al., 2009), (Marzio et al., 2011), (Oliveira et al., 2012), (Amiri et al., 2017), (Tsuruta and Sagae, 2017) and (Bedouhene and Zougab, 2019).

The multiplicative bias correction (MBC) technique improved the kernel estimation by reducing the order of magnitude of the bias from $O(\nu^{-1})$ to $O(\nu^{-2})$. Notice that the MBC techniques were originally proposed in linear symmetric and asymmetric kernel density estimation by (Terrell and Scott, 1980) (TS estimator) and (Jones et al., 1995) (JLN estimator). This techniques are later used by several authors: (Hirukawa, 2010; Hirukawa and Sakudo, 2014; Zougab and Adjabi, 2016; Funke and Kawka, 2015; Zougab et al., 2018) in continuous situation. (Harfouche et al., 2018) and (Harfouche et al., 2020) in discrete case. Recently, (Bedouhene and Zougab, 2019) applied the MBC techniques on the circular data which made it possible to improve the quality of the estimate by reducing the order of magnitude of bias from $O(\nu^{-1})$ to $O(\nu^{-2})$.

An alternative to nonparametric method, is the semiparametric estimate of the probability density proposed by (Hjort and Glad, 1995) (HG), using symmetric kernels for linear case. The idea of the latter, is to develop an estimator composed of two parts, the first one is the parametric, and the second represents the non-parametric correction function, this allows to modify the bias and keeps the variance; see also (Hagmann and Scaillet, 2007) using asymmetric kernels and (Kokonendji et al., 2009) using discrete kernels.

The present paper mainly focuses on two objectifs. The first objective is to investigate the semiparametric estimator on circular data and to develop the associated properties by using the Taylor series approximations. This has not yet been done in the literature. This work will give an idea on the efficiency of the semiparametric estimator compared to the nonparametric estimator on circular data. The semiparametric estimator given by

$$\tilde{f}(\theta) := g(\theta)\hat{r}(\theta), \theta \in [0, 2\pi), \tag{1}$$

where $g(\theta)$ is an initial density estimator and $\hat{r}(\theta) = \hat{f}(\theta)/g(\theta)$ is the correction factor. Based on the estimator defined by (1), several cases can be obtained by changing the function g, this allows us to extend the study to a more general study that includes several modes of estimating the probability density in the case of circular data. The function g can takes several forms: the uniform distribution $g \equiv 1/2\pi$ which gives a classical kernel estimator, the kernel estimator $g = \hat{f}$, which gives a bias-corrected estimator JLN (Jones et al., 1995), and a parametric model $g = f_{par}$ which gives a semiparametric estimator defined by $\tilde{f}_{HG}(\theta) = f_{par}(\theta)\hat{r}(\theta)$ (see,(Hjort and Glad, 1995),(Hirukawa and Sakudo, 2019) in the linear case).

To improve the quality of the semiparametric estimators HG, (Jones et al., 1999) proposed the so-called JSH estimator using the classical symmetric kernels. (Jones et al., 1999) are based on the same idea as (Hjort and Glad, 1995) but in the total nonparametric mode, that we correct the parametric part by a nonparametric correction what gives generally a same variance and a smaller bias. The principle of this estimator is to replace the function g by a semiparametric estimator $g = \tilde{f}_{HG}$. More recently, (Hirukawa and Sakudo, 2019) extends this idea to the asymmetric kernel density estimator with support $\mathbb{R}^+$. This study has shown that the JSH estimator reduces the order of magnitude of the bias from the $O(h^{-1})$ to $O(h^{-2})$ (h is the bandwidth parameter), which can even become unbiased under the right conditions.

The second objective of the paper is to improve the semiparametric HG ($\tilde{f}_{HG}$) estimator with von Mises kernel in the case of circular data by applying the JSH estimator, which will improve the order of magnitude of the bias from $O(\nu^{-1})$ of the HG estimator to $O(\nu^{-2})$ in the case of the JSH estimator and will keeps the order of magnitude of the variance. The asymptotic and global properties for the proposed JSH estimator are established.

This paper is organized as follows. Section 2, briefly recalls the classical von Mises kernel density and its properties. Section 3, presents the semiparametric von Mises kernel density estimator with different forms of the parametric part. In Section 4, we introduce the HG estimator and its properties for circular data. Section 5, develops the JSH estimator and its properties for circular case. The unbiased cross validation (UCV) procedure is adapted for choosing the optimal bandwidth. We examine the performance of the JSH estimator using data generated from known circular distributions via the integrated squared error (ISE) and integrated squared bias (ISB) criterions in section 6. Section 7, illustrates an application of real data. Section 8, concludes the paper.

2 von Mises kernel density estimators (A brief review)

Let $\Theta_1, \Theta_2, \ldots, \Theta_n$ be independant observation from a count distribution with unknown probability density function (pdf) f defined on the support $[0, 2\pi)$. A von Mises circular kernel density estimator for f can be expressed as (see for example (Taylor, 2008)):

$$\begin{aligned} \hat{f}_{vM}(\theta;\nu) &= \frac{1}{n}\sum_{i=1}^{n} k_\nu(\theta-\Theta_i) \\ &= \frac{1}{n2\pi I_0(\nu)}\sum_{i=1}^{n}\exp\{\nu\cos(\theta-\Theta_i)\}, 0 \le \theta < 2\pi, \end{aligned} \tag{2}$$

where $\theta > 0$ is the estimation point (angle where the density is estimated), $\nu = \nu(n) > 0$ is the smoothing parameter that fulfills $\lim_{n\to\infty} \nu(n) = 0$, and $I_z(.)$ denotes the modified bessel function of order z. The asymptotic formulas of bias and variance are given by (Taylor, 2008)):

$$Bias(\hat{f}_{vM}(\theta;\nu)) = \frac{1}{4\nu} f''(\theta) + o(\nu^{-1}),$$

and

$$Var(\hat{f}_{vM}(\theta;\nu)) = \frac{\nu^{\frac{1}{2}}}{2n\sqrt{\pi}} f(\theta) + o\left(\frac{\nu^{\frac{1}{2}}}{n}\right).$$

The next section discuss the semiparametric approach for circular kernel density estimation.

3 Semiparametric von Mises kernel density estimator

In this section, we present the semiparametric estimator which is composed of two parts parametric ($g(x)$) and non parametric ($\hat{r}(x)$). Let $\Theta_1, \Theta_2, \ldots, \Theta_n$ be independant observation from a count distribution with unknown probability density function (pdf) f defined on the support $[0, 2\pi)$. The semiparametric kernel density estimator $\tilde{f}$ is given by:

$$\tilde{f}(\theta) := g(\theta)\hat{r}(\theta) := g(\theta)\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{k_\nu(\theta-\Theta_i)}{g(\Theta_i)}\right\}, \tag{3}$$

where g is the function that can be take many forms (constant, estimator density, parametric and semiparametric model) and $\hat{r}(x)$ serves as a correction factor. The form of the semiparametric von

Mises kernel density estimator that depends on a design point θ and smoothing parameter ν is as following:

$$\tilde{f}_{vM}(\theta) := g(\theta)\hat{r}(\theta) := g(\theta)\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{g(\Theta_i)}\right\}, \quad (4)$$

The estimator $\tilde{f}$ changes with the change of the function g, several cases of estimators can be derived according to the function g form.

3.1 Function g is an uniform density

When the function g is an uniform density i.e $g(\theta) \equiv 1/2\pi$, the estimator $\tilde{f}_{vM}$ is reduced to a classical estimator $\tilde{f}_{vM,Classic}$ defined by (2), as shown below:

$$\begin{aligned}
\tilde{f}_{vM}(\theta) &= g(\theta)\hat{r}(\theta), \\
&= g(\theta)\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{g(\Theta_i)}\right\}, \\
&= \left\{\frac{1}{2\pi}\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{\frac{1}{2\pi}}\right\}, \\
&= \frac{1}{n2\pi I_0(\nu)}\sum_{i=1}^{n}\exp\{\nu\cos(\theta-\Theta_i)\}, \\
&= \tilde{f}_{vM,Classic}(\theta). \quad (5)
\end{aligned}$$

The properties of $\tilde{f}_{vM,Classic}$ given by (5) are developed by (Taylor, 2008), (Marzio et al., 2009, 2011).

3.2 Function g is a kernel density estimator

When the function g is a kernel density estimator i.e $g(\theta) = \hat{f}_{vM}(\theta)$, the estimator $\tilde{f}_{vM}$ becomes the (Jones et al., 1995), type fully nonparametric MBC estimator (JLN) $\tilde{f}_{vM,JLN}$ given by:

$$\tilde{f}_{vM,JLN}(\theta) = \hat{f}_{vM}(\theta)\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{\hat{f}_{vM}(\Theta_i)}\right\}. \quad (6)$$

This estimator is introduced by (Jones et al., 1995), and considered by (Hirukawa, 2010), (Hirukawa and Sakudo, 2014) for asymmetric kernel density estimation, (Zougab and Adjabi, 2016) for heavy tailed data using generalized Birnbaum-Saunders kernels, (Harfouche et al., 2018) and (Harfouche et al., 2020) for discrete situations. More recently, (Bedouhene and Zougab, 2019) have examined the JLN-von Mises kernel estimator for circular data.

3.3 Function g is a parametric model

When g belongs to a parametric family, the function g can takes different circular parametric models f_{par} (von Mises, Projected Normal, Wrapped distributions, ...), then $\tilde{f}_{vM}$ reduces to the (Hjort and Glad, 1995) type semiparametric MBC estimator $\tilde{f}_{vM,HG}$, studied by (Hagmann and Scaillet, 2007) using asymmetric kernel and by (Kokonendji et al., 2009) for discrete data. Note that the HG-von Mises circular kernel estimator and its properties will be presented in section 4.

3.4 Function g is semiparametric estimator $\tilde{f}_{vM,HG}$

In this case, (Jones et al., 1999) proposed a new JSH estimator which improves the quality of the estimator as well as the bias compared to the HG estimators already mentioned.

The JSH estimator principle, is to consider the semiparametric $\tilde{f}_{vM}$ estimator and replace the function g by the $\tilde{f}_{vM,HG}$ estimator, which improves the convergence of the bias towards $o(\nu^{-2})$ whatever the scenario. The section 5, discusses in details the estimator and its properties.

4 The HG estimator and its properties

The semiparametric HG von Mises kernel estimator $\tilde{f}_{vM,HG}$ is given as follows:

$$\tilde{f}_{vM,HG}(\theta) = f(\theta;\hat{\vartheta})\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{f(\Theta;\hat{\vartheta})}\right\}. \tag{7}$$

Where $f(\theta;\vartheta)$ is the pdf of the von Mises distribution $vM(\vartheta)$ with $\vartheta = (\mu, k)$. the von Mises distribution is expressed as:

$$f(\theta;\mu,k) = \frac{1}{2\pi I_0(k)}\exp\left(k\cos(\theta-\mu)\right), \;\; 0 \le \theta < 2\pi,$$

and $\hat{\vartheta} = (\hat{\mu}, \hat{k})$ is the estimated parameter of the $\vartheta = (\mu, k)$ by the maximum likelihood (ML) method.

4.1 Asymptotic properties

To approximate the bias and variance of HG-vM kernel estimator, we assume that:
Set $\vartheta_0 = (\mu_0, k_0)$ the value which minimizes the Kullback–Leibler distance of $f(\theta;\vartheta)$ from the true $f(\theta)$, we also denote $f_0(\cdot) = f(\cdot, \vartheta_0)$,

A1. f has four continuous and bounded derivatives.
A2. The sequence of bandwidths $\nu = \nu(n)$, satisfies $\nu \to \infty$ and $\nu^{1/2}/n \to 0$ when $n \to \infty$

Theorem 4.1. *Under the Assumptions 1-2, the bias and variance of* HG-vM *kernel estimator defined by (7), for a given $\theta \in [0, 2\pi)$, are given by:*

(i) The bias of HG-vM *kernel estimator is given by:*

$$Bias\left(\tilde{f}_{vM,HG}(\theta)\right) = \frac{1}{4\nu}f(\theta;\vartheta_0)r''(\theta) + o(\nu^{-1}), \tag{8}$$

(ii) The variance of HG-vM *kernel estimator is given by:*

$$Var\left(\tilde{f}_{vM,HG}(\theta)\right) = \frac{\nu^{1/2}}{2n\sqrt{\pi}}f(\theta) + o\left(\frac{\nu^{1/2}}{n}\right). \tag{9}$$

Proof. *A Taylor expansion gives*

$$\begin{aligned}
\frac{f(\theta;\widehat{\vartheta})}{f(\Theta_i;\widehat{\vartheta})} &= \exp\{\log f(\theta;\widehat{\vartheta}) - \log f(\Theta_i;\widehat{\vartheta})\} \\
&\doteq \frac{f_0(\theta)}{f_0(\Theta_i)} + \frac{f_0(\theta)}{f_0(\Theta_i)}\{u_0(\theta) - u_0(\Theta_i)\}^{\mathsf{T}}\left(\widehat{\vartheta} - \vartheta_0\right) \\
&= \frac{f_0(\theta)}{f_0(\Theta_i)}\left[1 - \{u_0(\Theta_i) - u_0(\theta)\}^{\mathsf{T}}\left(\widehat{\vartheta} - \vartheta_0\right)\right],
\end{aligned}$$

where $u_0(\theta) = \partial \log f(\theta;\vartheta_0)/\partial\vartheta$. *The semiparametric estimator (7) can be approximated as follows:*

$$\widehat{f}_{\nu}^{HG-VM}(\theta) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}\frac{f_0(\theta)}{f_0(\Theta_i)}\left[1-\{u_0(\Theta_i)-u_0(\theta)\}^{\intercal}\left(\widehat{\vartheta}-\vartheta_0\right)\right]. \tag{10}$$

Using the representation (10), the properties of the von Mises random variable and the Taylor development when $\nu \to \infty$ *(see (Mardia and Jupp, 2009)) and the same computational steps of (Hjort and Glad, 1995), we obtain the results given in the theorem 4.1.* □

Note that the asymptotic variance of the HG-VM estimator is similar to that of the classical von Mises estimator.

Corollary 1. *The criterion to use for the global property is the mean integrated squared error* (MISE) *defined as:*

$$\begin{aligned} MISE\left(\tilde{f}_{vM,HG}(\theta)\right) &= \int_0^{2\pi}\mathbb{E}\left(\tilde{f}_{vM,HG}(\theta)-f(\theta)\right)^2 d\theta \\ &= \int_0^{2\pi} MSE\tilde{f}_{vM,HG}(\theta)d\theta \\ &= \int_0^{2\pi} bias^2\left(\tilde{f}_{vM,HG}(\theta)\right)d\theta + \int_0^{2\pi} Var\left(\tilde{f}_{vM,HG}(\theta)\right)d\theta \\ &= \frac{1}{16\nu^2}\int_0^{2\pi}\{f_0(\theta)r''(\theta)\}^2 d\theta + \frac{\nu^{1/2}}{2n\sqrt{\pi}} + o\left(\nu^{-2}+\frac{\nu^{1/2}}{n}\right). \end{aligned} \tag{11}$$

By minimizing (11) in the bandwidth ν, *we obtain the optimal value:*

$$\nu^{*}_{vM,HG} = \left\{\frac{n\sqrt{\pi}}{2}\int_0^{2\pi}\{f_0(\theta)r''(\theta)\}^2 d\theta\right\}^{2/5}. \tag{12}$$

The optimal $MISE^{*}_{vM,HG}$ *of HG estimator is obtained by remplacing the* $\nu^{*}_{vM,HG}$ *given by (12) in* $MISE\left(\tilde{f}_{vM,HG}(\theta)\right)$ *given in (11)*

$$MISE^{*}\left(\tilde{f}_{vM,HG}(\theta)\right) = \frac{1}{2\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\int_0^{2\pi}\{f_0(\theta)r''(\theta)\}^2 d\theta\right)^{1/5} \left(1+\frac{1}{4}\left\{\int_0^{2\pi}\{f_0(\theta)r''(\theta)\}^2 d\theta\right\}^{-1}\right)n^{-4/5}.$$

5 The JSH estimator

Based on the same idea of (Jones et al., 1999) and (Hirukawa and Sakudo, 2019), JSH-von Mises kernel density estimator will be defined as follows:

$$\tilde{f}_{vM,JSH}(\theta) = \tilde{f}_{vM,HG}(\theta)\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{\tilde{f}_{vM,HG}(\Theta_i)}\right\}, \tag{13}$$

where $\tilde{f}_{vM,HG}(\theta) = f(\theta)\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{f(\Theta_i)}\right\}$,
$f(\theta)$ is the parametric model, which can takes different distributions. In the case where this model takes the form of a uniform distribution, the JSH estimator is reduced into an MBC JLN estimator, which explains why the JLN estimator is a special case of the general estimator JSH.
In this work, the parametric model exploited is the von-Mises law of parameters μ and k, expressed as:

$$f(\theta;\mu,k) = \frac{1}{2\pi I_0(k)}\exp\left(k\cos(\theta-\mu)\right),\ \ 0 \leq \theta < 2\pi,$$

and $\hat{\mu}$ and $\hat{k}$ are the estimated parameters of the μ and k respectively by the maximum likelihood (ML) method.

5.1 Asymptotic properties

The asymptotic properties of the JSH-vM kernel estimator requires certain conditions and assumptions.

Set μ_0 and k_0 the values which minimizes the Kullback–Leibler distance of $f(\theta;\mu,k)$ from the true $f(\theta)$. We also denote $r_0(.) = f(.)/f(.;\mu_0,k_0)$

A3. $\{\Theta_i\}_{i=1}^{n}$ are i.i.d random variables drown from a univariate distribution having a density f with support $[0,2\pi)$.

A4. For a given design point $\theta \in [0,2\pi)$, $f(\theta)$, $f(\theta;\mu_0,k_0) > 0$, and $r_0(\theta)$ has four continuous and bounded derivatives in the neighborhood of θ.

Note that these assumptions have been discussed in (Hirukawa and Sakudo, 2019).

Theorem 5.1. *Under the Assumptions 1-4, the bias and variance of* JSH-vM *kernel estimator defined by (5.1), for a given $\theta \in [0,2\pi)$, are given by:*

(i) The bias of JSH-vM *kernel estimator is given by:*

$$Bias\left(\tilde{f}_{vM,JSH}(\theta)\right) = -f(\theta)\,q\left(\theta,r_0\right)\,\frac{1}{\nu^2} + o\left(\frac{1}{\nu^2}\right), \tag{14}$$

where $q(\theta,r_0) = \left(\frac{r_0''(\theta)}{4r_0(\theta)}\right)''$.

(ii) The variance of JSH-vM *kernel estimator is given by:*

$$Var\left(\tilde{f}_{vM,JSH}(\theta)\right) = \frac{\nu^{1/2}}{2n\sqrt{\pi}}f(\theta) + o\left(\frac{\nu^{1/2}}{n}\right). \tag{15}$$

Proof. *It is easy to show that when $\nu \longrightarrow \infty$ and $\nu^{1/2}/n \longrightarrow 0$ when $n \longrightarrow \infty$,*

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{g(\Theta_i;\widehat{\vartheta})}\right] \simeq r(\theta) + \frac{1}{4\nu}r''(\theta), \tag{16}$$

and

$$\mathbb{V}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{2\pi I_0(\nu)}\exp\{\nu\cos(\theta-\Theta_i)\}}{g(\Theta_i;\widehat{\vartheta})}\right]\simeq\frac{\nu^{1/2}}{2n\sqrt{\pi}}\frac{r(\theta)}{g_0(\theta)}. \tag{17}$$

where $\simeq$ means asymptotically equal.
Combining the proof of Theorem 2 proposed by (Bedouhene and Zougab, 2019) and the approximations (16) and (17), we establish the results of Theorem 5.1. □

Corollary 2. *The criterion to use for the global property is the mean integrated squared error* (MISE)*, defined as:*

$$\begin{aligned} MISE\left(\tilde{f}_{vM,JSH}(\theta)\right) &= \int_0^{2\pi} bias^2\left(\tilde{f}_{vM,JSH}(\theta)\right)d\theta + \int_0^{2\pi} Var\left(\tilde{f}_{vM,JSH}(\theta)\right)d\theta \\ &= \frac{1}{\nu^4}\mathbb{E}\left[f(\Theta)\,q^2(\Theta,r_0)\right]+\frac{\nu^{1/2}}{2n\sqrt{\pi}}+o\left(\nu^{-4}+\frac{\nu^{1/2}}{n}\right). \end{aligned} \tag{18}$$

The optimal bandwidth minimizing the corresponding MISE *(18) is such that*

$$\nu^*_{JSH}=\left\{16\,n\,\sqrt{\pi}\mathbb{E}\left[f(\Theta)\,q^2(\Theta,r_0)\right]\right\}^{2/9}. \tag{19}$$

Therefore, the optimal MISE *becomes*

$$MISE^*\left(\tilde{f}_{vM,JSH}(\theta)\right)=\frac{1}{(16\sqrt{\pi})^{8/9}}\ \left\{\mathbb{E}\left[f(\Theta)\,q^2(\Theta,r_0)\right]\right\}^{1/9}n^{-8/9}. \tag{20}$$

In practice, the bandwidth parameter ν^*_{JSH} cannot be employed because it's depend on the unknown density f and its derivatives. For this, we present in the next section the unbiased cross validation UCV method for selecting the smoothing parameter.

6 Bandwidth choice by UCV method

We adopt in this section the popular unbiased cross validation (UCV) method introduced by (Rudemo, 1982) and (Bowman, 1984), and recently applied by several authors (See (Zougab and Adjabi, 2016; Harfouche et al., 2018, 2020) and (Bedouhene and Zougab, 2019) for the MBC technique, (Hagmann and Scaillet, 2007) for the semiparametric MBC technique). The optimal bandwidth ν by UCV method for a given estimator $\tilde{f}_{vM,JSH}$ is obtained by:

$$\nu_{UCV}=\arg\min_{\nu} UCV(\nu),$$

where

$$\begin{aligned} UCV(\nu) &= \frac{1}{n^2}\int_0^{2\pi}\tilde{f}^2_{vM,HG}(\theta)\left(\sum_{i=1}^{n}\frac{k_\nu(\theta-\Theta_i)}{\tilde{f}_{vM,HG}(\Theta_i)}\right)^2 d\theta \\ &- \frac{2}{n(n-1)}\sum_i\sum_{j\neq i}k_\nu(\theta-\Theta_i)\frac{\tilde{f}_{vM,HG}(\Theta_i)}{\tilde{f}_{vM,HG}(\Theta_j)}. \end{aligned} \tag{21}$$

7 Simulation study

This part is devoted to a simulation study, which consists to evaluate the performance of the $\tilde{f}_{vM}$, $\tilde{f}_{vM-JLN}$, $\tilde{f}_{vM-HG}$ and $\tilde{f}_{vM-JSH}$ estimators. This simulation study is based on 100 replications for samples sizes $n = 10$, 50, 100 and 200 for 20 circular models (distributions), that were used by (Oliveira et al., 2012), (García-Portugués, 2013) and recently (Bedouhene and Zougab, 2019, 2020). The models are classified into four sets, each with its complexity:

1. Simple models: circular uniform (M1); von Mises (M2); wrapped Normal (M3); cardioid (M4); wrapped Cauchy (M5) and wrapped skew-Normal (M6).
2. Two components models: von Mises mixtures (M7, M8 and M9); mixture of von Mises and wrapped Cauchy (M10).
3. Models with more than two components: von Mises mixtures with three components (M11, M12 and M13); von Mises mixture with four components (M14); mixture of wrapped Cauchy, wrapped Normal, von Mises and wrapped skew-Normal (M15); von Mises mixture with five components (M16).
4. Other complex models: mixture of cardioid and wrapped Cauchy (M17); mixture of von Mises (M18 and M19); mixture of two wrapped skew-Normal and two wrapped Cauchy (M20).

For the comparison, we used the classical, JLN, HG and JSH vM kernel estimators and the UCV method for bandwidth parameter selection. Note that, The parametric part $f(\theta; \mu, k)$ of the $\tilde{f}_{HG,vM}$ estimator is considered as vM distribution of μ and k parameters $vM(\mu, k)$. The parameters μ and k were estimated by maximum likelihood (ML). We examine the performances of the estimators via integrated squared error (ISE) and integrated squared bias (ISB) given respectively by:

$$ISE := \int_0^{2\pi} \left(\tilde{f}(\theta) - f(\theta)\right)^2 d\theta \tag{22}$$

and

$$ISB := \int_0^{2\pi} \left(\mathbb{E}(\tilde{f}(\theta)) - f(\theta)\right)^2 d\theta \tag{23}$$

Tables 1, 2, 3 and 4 show the average ISE of the vM kernel estimators. We can observe that,

- The means of ISE decrease as sample size n increases for the all estimators, which indicates that our estimators are consistent.
- The vM-JLN estimator performs better than the other competitors for models M5, M6, M7, M10, M11, M12, M13, M14, M16, M17, M18 and M20 and for a samples size $n = 100$ and $n = 200$ except model M10 for $n = 100$.
- The vM-JLN estimator performs better than the other estimators for models M6, M7, M8, M11, M13, M14, M16 and M17 and for a sample size $n = 50$.
- For a sample size $n = 10$ the standard vM kernel estimator is more efficient for the models M1, M4, M10, M11, M12, M16, M18, M19 and M20.
- For the rest of models M2, M3, M9 and M15 the perfomance of estimators are mixed depending on the sample size.

Tables 5 and 6 show the average ISB of the vM kernel estimators. We can observe that:

Table 1: Average integrated squared error ($\overline{ISE}$) and their standard deviation between parentheses based on 100 replications for 20 models with sample size $n = 10$

$n = 10$	$\tilde{f}_{vM}$ (Sd_{vM})	$\tilde{f}_{vM-JLN}$ (Sd_{vM-JLN})	$\tilde{f}_{vM-HG}$ (Sd_{vM-HG})	$\tilde{f}_{vM-JSH}$ (Sd_{vM-JSH})
M1	**0.028504** (0.040126)	0.044624 (0.048535)	0.085598 (0.042796)	0.065051 (0.046547)
M2	0.044730 (0.027727)	0.054532 (0.038296)	**0.041635** (0.032852)	0.056422 (0.043456)
M3	0.080912 (0.063078)	0.077513 (0.045428)	0.066713 (0.048654)	**0.061435** (0.029901)
M4	**0.031643** (0.018711)	0.035378 (0.020203)	0.032978 (0.030695)	0.036326 (0.021759)
M5	0.175951 (0.039888)	0.148754 (0.050878)	0.165277 (0.046911)	**0.136418** (0.050530)
M6	0.081913 (0.030951)	0.076863 (0.024004)	**0.071529** (0.019551)	0.075294 (0.019237)
M7	0.056374 (0.024210)	**0.054870** (0.019606)	0.067062 (0.040294)	0.057183 (0.026707)
M8	0.073435 (0.034446)	0.065588 (0.031154)	0.068817 (0.030425)	**0.062101** (0.033644)
M9	0.038088 (0.022375)	**0.037386** (0.022005)	0.058252 (0.029270)	0.043407 (0.026101)
M10	**0.101301** (0.050482)	0.107924 (0.057971)	0.104519 (0.058812)	0.109033 (0.062352)
M11	**0.051501** (0.012703)	0.057676 (0.014236)	0.069096 (0.033663)	0.067695 (0.032629)
M12	**0.066814** (0.038738)	0.074644 (0.042631)	0.101371 (0.039642)	0.104655 (0.040974)
M13	0.097510 (0.034632)	0.096302 (0.034424)	0.126583 (0.067811)	**0.095344** (0.046333)
M14	0.093109 (0.021393)	**0.081828** (0.030255)	0.116401 (0.029262)	0.091273 (0.034697)
M15	0.018991 (0.018817)	0.025672 (0.026108)	0.071526(0.068303)	0.066495 (0.051130)
M16	**0.085333** (0.014261)	0.090423 (0.014995)	0.134788 (0.055393)	0.108673 (0.028095)
M17	0.135170 (0.053391)	0.119433 (0.044483)	**0.111107** (0.019398)	0.121193 (0.047785)
M18	**0.074270** (0.037493)	0.076313 (0.037901)	0.102535 (0.105780)	0.092129 (0.076978)
M19	**0.080081** (0.027177)	0.085918 (0.036308)	0.090258 (0.024831)	0.094315 (0.035757)
M20	**0.121919** (0.029699)	0.127653 (0.039151)	0.150788 (0.047860)	0.143430 (0.040143)

- For all estimators, the means of ISB based on 100 simulations decrease as sample size n increases.
- For all samples size the vM-JLN and vM-JSH kernel estimators outperform the classical vM and vM-HG kernel estimators except models (M1, M2, M4, M10 and M12 for $n = 10$), (M1, M2, M4 and M15 for $n = 50$) and (M1 and M2 for $n = 100$ et 200).
- The performance of vM-JLN and vM-JSH kernel estimators are mixed depending on the models, this is explained by the fact that the vM-JLN estimator is a special case of vM-JSH kernel estimators

The figure 1 presents the plot of pdf M3, M8, M11 and M17 models for each distribution family using vM, JLN-vM, HG-vM and JSH-vM estimators with UCV approach for bandwidth choice based on sample size $n = 200$ for one application. The plots show that in general the smoothing quality is satisfactory for all models. The estimators were able to reproduce the uni and several modes of the considered models, except for the model 17.

8 Illustration with real data

In this section, we illustrate the application of $\tilde{f}_{vM}$, $\tilde{f}_{vM-JLN}$, $\tilde{f}_{vM-HG}$ and $\tilde{f}_{vM-JSH}$ estimators in practice, we have analysed two real data sets.

Exemple 1. (Time series of flare azimuths): this data present a time series of measurements obtained from an experiment, to assess the relative stability of flare-projectile assemblies. A flare, attached to a projectile, is launched upward from a launch point O in a fixed direction. At some

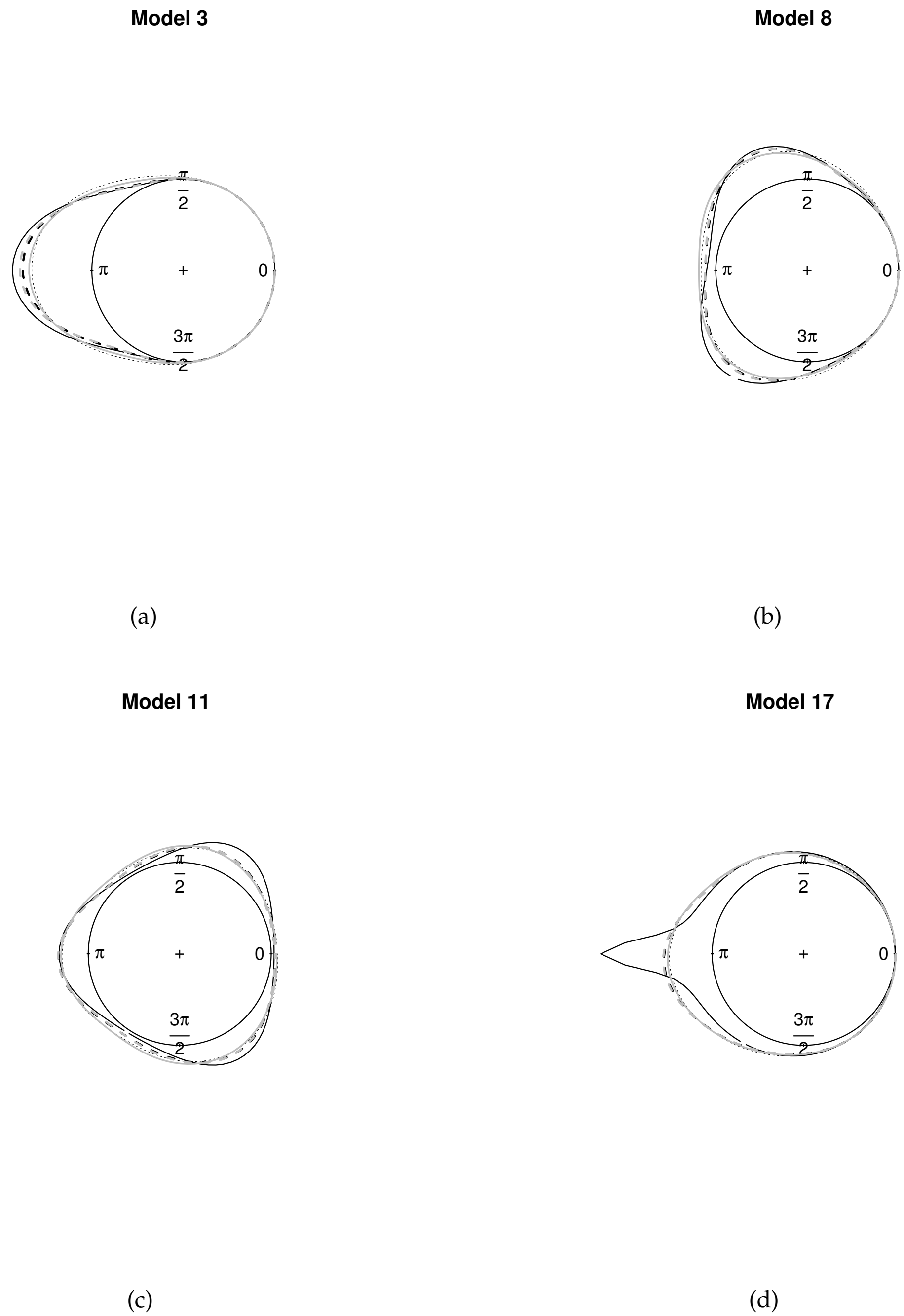


Figure 1: True pdf and vM kernel estimators for models 3, 8, 11 and 17 with $n = 200$. Black solid line: true density; Black dashed line: vM estimator; Black Dotted line: vM-JLN estimator; Grey solide line: vM-HG estimator and Grey dashed line: vM-JSH estimator.

Table 2: Average integrated squared error ($\overline{ISE}$) and their standard deviation between parentheses based on 100 replications for 20 models with sample size $n = 50$

$n = 50$	$\tilde{f}_{vM}$ (Sd_{vM})	$\tilde{f}_{vM-JLN}$ (Sd_{vM-JLN})	$\tilde{f}_{vM-HG}$ (Sd_{vM-HG})	$\tilde{f}_{vM-JSH}$ (Sd_{vM-JSH})
M1	**0.001866** (0.003489)	0.006231 (0.005917)	0.013917 (0.005010)	0.011227 (0.005081)
M2	0.010313 (0.008561)	0.011044 (0.009943)	**0.008018** (0.006045)	0.011511 (0.009034)
M3	0.041835 (0.014696)	0.017860 (0.008250)	0.028309 (0.011004)	**0.014193** (0.007108)
M4	**0.008945** (0.008045)	0.014548 (0.011967)	0.014283 (0.015017)	0.018999 (0.013630)
M5	0.154229 (0.019441)	0.109994 (0.028114)	0.139815 (0.017119)	**0.103537** (0.029125)
M6	0.058575 (0.009952)	**0.047698** (0.011597)	0.048187 (0.006382)	0.051759 (0.016907)
M7	0.019584 (0.007133)	**0.014356** (0.006791)	0.022576 (0.005455)	0.014708 (0.007069)
M8	0.026306 (0.012476)	**0.018937** (0.014137)	0.034317 (0.021504)	0.020011 (0.017085)
M9	0.009666 (0.006145)	0.009945 (0.004733)	0.013562 (0.007814)	**0.009470** (0.004791)
M10	0.052599 (0.011198)	0.042107 (0.013285)	0.044380 (0.010356)	**0.042057** (0.014870)
M11	0.031501 (0.009060)	**0.024606** (0.010623)	0.036343 (0.011329)	0.025054 (0.010273)
M12	0.017454 (0.011790)	0.015778 (0.007861)	0.020647 (0.010969)	**0.014755** (0.005718)
M13	0.042008 (0.006729)	**0.031182** (0.006917)	0.048533 (0.008577)	0.032526 (0.007163)
M14	0.053977 (0.007150)	**0.040523** (0.007811)	0.064132 (0.015627)	0.043959 (0.010567)
M15	**0.009173** (0.004106)	0.012301 (0.006091)	0.019124 (0.006767)	0.017794 (0.008228)
M16	0.074266 (0.007694)	**0.062254** (0.009584)	0.074199 (0.008923)	0.063870 (0.010247)
M17	0.081349 (0.008914)	**0.071710** (0.011465)	0.078328 (0.011586)	0.071803 (0.013159)
M18	0.040710 (0.005906)	0.042476 (0.006280)	**0.040324** (0.005152)	0.043975 (0.007533)
M19	0.035764 (0.006904)	**0.035098** (0.008540)	0.041890 (0.014962)	0.035811 (0.009832)
M20	0.080598 (0.005963)	**0.072905** (0.007403)	0.086780 (0.005067)	0.075472 (0.007073)

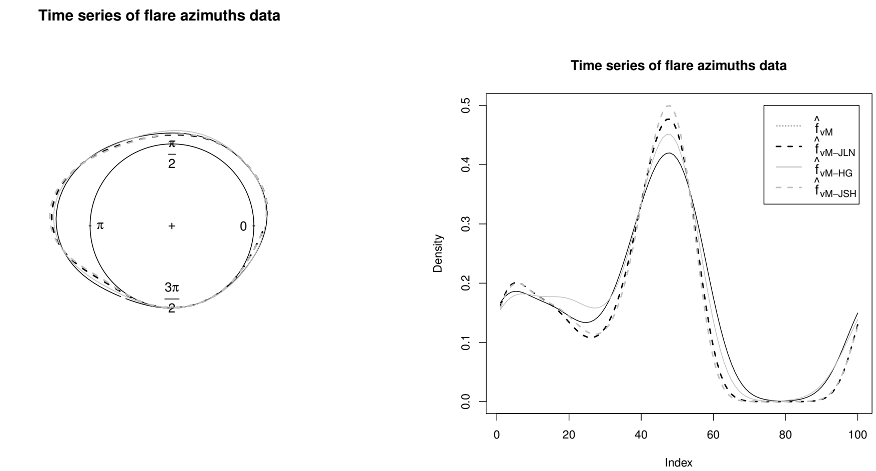
point P in space, the flare commences burning. The azimuth of P relative to O gives an indication of the variability of the assembly as more and more trials are conducted with it. The data shown are based on 60 successive launches, (see, (Fisher, 1995)).

Exemple 2. (Long-axis orientations of feldspar laths): a data of 133 measurements of feldspar laths in basalt reported by (Smith, 1988) and presented by (Fisher, 1995).

For analyzing these data sets, we applied four estimators $\tilde{f}_{vM}$, $\tilde{f}_{vM-JLN}$, $\tilde{f}_{vM-HG}$ and $\tilde{f}_{vM-JSH}$ to estimate the probability density function with vM circular kernel. The UCV technique is employed for bandwidth choice. For the $vM-HG$ kernel estimator, the parameters μ and k of parametric model are estimated by ML method. The values of $\hat{\mu}$ and $\hat{k}$ are given in table 7.

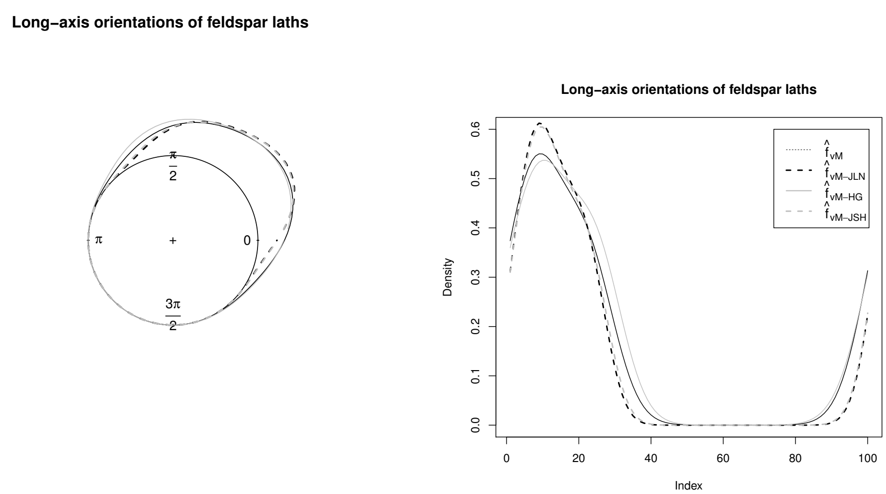
The figures 2 and 3 show the linear and circular estimators ($\tilde{f}_{vM}$, $\tilde{f}_{vM-JLN}$, $\tilde{f}_{vM-HG}$ and $\tilde{f}_{vM-JSH}$) for Time series of flare azimuths data with sample size $n = 60$ and Long-axis orientations of feldspar laths data with sample size $n = 133$, where the solide and dashed lines in black represent the $\tilde{f}_{vM}$ and $\tilde{f}_{vM-JLN}$ estimators, the solide and dashed lines in grey represent $\tilde{f}_{vM-HG}$ and $\tilde{f}_{vM-JSH}$ estimators respectively.

From the figures 2 and 3, we can see that all estimators are capable of reproducing the unimodality of these data sets. We also note that the smoothing quality is satisfactory and almost similar for the four estimators except in certain regions.



(a) Linear plot for Time series of flare azimuths data.
(b) Circular plot for Time series of flare azimuths data.

Figure 2: vM kernel estimator for Time series of flare azimuths data with sample size $n = 60$. Black solid line: vM estimator; Black dashed line: JLN-vM estimators; Grey solid line: HG-vM; Grey dashed line: JSH-vM.



(a) circular plot for Long-axis orientations of feldspar laths data.
(b) linear plot for Long-axis orientations of feldspar laths data.

Figure 3: vM kernel estimator for Long-axis orientations of feldspar laths data with sample size $n = 133$. Black solid line: vM estimator; Black dashed line: JLN-vM estimators; Grey solid line: HG-vM; Grey dashed line: JSH-vM.

Table 3: Average integrated squared error ($\overline{ISE}$) and their standard deviation between parentheses based on 100 replications for 20 models with sample size $n = 100$

$n = 100$	$\tilde{f}_{vM}$ (Sd_{vM})	$\tilde{f}_{vM-JLN}$ (Sd_{vM-JLN})	$\tilde{f}_{vM-HG}$ (Sd_{vM-HG})	$\tilde{f}_{vM-JSH}$ (Sd_{vM-JSH})
M1	**0.001457** (0.002025)	0.003106 (0.003671)	0.007134 (0.002430)	0.008218 (0.002656)
M2	0.007618 (0.007692)	0.006788 (0.005903)	**0.005093** (0.005007)	0.006385 (0.005543)
M3	0.037664 (0.003969)	0.013694 (0.002814)	0.026014(0.004538)	**0.010504** (0.003361)
M4	**0.006219** (0.004247)	0.006670 (0.005332)	0.006363 (0.003905)	0.008482 (0.005162)
M5	0.154077 (0.017058)	**0.106388** (0.017568)	0.138480 (0.016540)	0.119237 (0.041765)
M6	0.057178 (0.010608)	**0.047702** (0.016986)	0.047161 (0.007705)	0.052374 (0.019852)
M7	0.017531 (0.008879)	**0.010608** (0.006796)	0.021808 (0.011133)	0.011076 (0.007525)
M8	0.021781 (0.005358)	**0.011615** (0.005049)	0.027436 (0.010461)	0.011731 (0.005784)
M9	0.006189 (0.003258)	0.005196 (0.002406)	0.007646 (0.004701)	**0.004875** (0.003331)
M10	0.042288 (0.008199)	0.033879 (0.005501)	0.036614 (0.006635)	**0.033876** (0.005801)
M11	0.023124 (0.002969)	**0.012971** (0.002866)	0.028574 (0.003982)	0.013945 (0.003178)
M12	0.010648 (0.005060)	**0.009277** (0.004555)	0.013961 (0.004866)	0.009663 (0.004217)
M13	0.040233 (0.005926)	**0.024880** (0.005162)	0.047879 (0.008810)	0.026472 (0.005187)
M14	0.048432 (0.006302)	**0.033535** (0.007966)	0.055748 (0.005895)	0.035982 (0.007658)
M15	**0.007635** (0.003075)	0.008174 (0.003857)	0.008748 (0.003335)	0.008276 (0.004931)
M16	0.064286 (0.003487)	**0.053544** (0.004683)	0.068856 (0.003156)	0.054935 (0.003064)
M17	0.079762 (0.006565)	**0.066913** (0.007029)	0.076166 (0.008493)	0.067919 (0.009618)
M18	0.035264 (0.005775)	**0.034528** (0.005536)	0.036797 (0.005094)	0.036755 (0.005704)
M19	0.029451 (0.005088)	0.026673 (0.004411)	0.030842 (0.007298)	**0.026669** (0.004697)
M20	0.072511 (0.004133)	**0.062957** (0.004733)	0.075288 (0.003609)	0.063519 (0.003887)

9 Conclusion

In this work, we extended the application of the semiparametric estimators HG and JSH, for the estimation of the probability density of circular data with von Mises kernel. We have shown that the JSH estimator improves the order of magnitude of the bias, whatever the parametric model chosen. We also evaluated the performance of the proposed estimator (JSH) by a comparative study with the classical estimator $\hat{f}_{vM}$, the estimator JLN $\tilde{f}_{JLN}$ and the semiparametric estimator $\tilde{f}_{HG}$. Our study showed that the proposed JSH estimator and the JLN estimator are more efficient than the other two estimators in terms of ISE and ISB. The two estimators JLN and JSH are almost similar, this is explained by the fact that the estimator JLN is a special case of JSH.

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Table 4: Average integrated squared error ($\overline{ISE}$) and their standard deviation between parentheses based on 100 replications for 20 models with sample size $n = 200$

$n = 200$	$\tilde{f}_{vM}$ (Sd_{vM})	$\tilde{f}_{vM-JLN}$ (Sd_{vM-JLN})	$\tilde{f}_{vM-HG}$ (Sd_{vM-HG})	$\tilde{f}_{vM-JSH}$ (Sd_{vM-JSH})
M1	**0.000840** (0.001765)	0.002012 (0.002658)	0.005313 (0.002595)	0.004798 (0.002703)
M2	0.005359 (0.004786)	0.004924 (0.003438)	**0.002080** (0.001720)	0.004924 (0.002783)
M3	0.033638 (0.006134)	0.010285 (0.002248)	0.021982 (0.004420)	**0.006965** (0.001297)
M4	0.003911 (0.001736)	0.003399 (0.002243)	**0.003174** (0.001561)	0.003761 (0.002169)
M5	0.147711 (0.003426)	**0.098190** (0.005221)	0.133369 (0.003885)	0.104548 (0.042274)
M6	0.055673 (0.005007)	**0.039312** (0.007453)	0.045489 (0.002868)	0.042482 (0.010633)
M7	0.014555 (0.003525)	**0.006889** (0.002733)	0.017779 (0.005106)	0.007382 (0.003266)
M8	0.020626 (0.002848)	0.009474 (0.002975)	0.023749 (0.006376)	**0.008934** (0.004090)
M9	0.005869 (0.003752)	0.004805 (0.003233)	0.006548 (0.003732)	**0.004541** (0.002573)
M10	0.040668 (0.005104)	**0.031890** (0.005037)	0.035392 (0.003461)	0.031992 (0.005620)
M11	0.022043 (0.003397)	**0.011385** (0.003413)	0.025781 (0.005939)	0.011667 (0.004046)
M12	0.007827 (0.001894)	**0.004541** (0.002231)	0.012173 (0.002798)	0.004888 (0.002422)
M13	0.036175 (0.003118)	**0.020665** (0.002707)	0.040786 (0.004169)	0.021381 (0.002714)
M14	0.046397 (0.002191)	**0.029773** (0.002206)	0.050799 (0.002620)	0.031039 (0.002747)
M15	0.006984 (0.002966)	0.006893 (0.003779)	0.006948 (0.004536)	**0.006829** (0.004414)
M16	0.061072 (0.002369)	**0.049232** (0.003386)	0.064606 (0.003178)	0.050812 (0.004028)
M17	0.079144 (0.007085)	**0.066755** (0.007067)	0.075603 (0.006609)	0.067226 (0.011314)
M18	0.034933 (0.003938)	**0.032185** (0.003686)	0.033361 (0.002792)	0.032942 (0.004285)
M19	0.028672 (0.002181)	0.023275 (0.001647)	0.030065(0.003587)	**0.023245** (0.001921)
M20	0.071069 (0.001973)	**0.060446** (0.002518)	0.072983 (0.001576)	0.060781 (0.001884)

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Table 5: Average integrated squared bias ($\overline{ISB}$) based on 100 replications for 20 models with samples size $n = 10$ and $n = 50$.

	$n = 10$				$n = 50$			
	ISB_{vM}	ISB_{vM-JLN}	ISB_{vM-HG}	ISB_{vM-JSH}	ISB_{vM}	ISB_{vM-JLN}	ISB_{vM-HG}	ISB_{vM-JSH}
M1	**0.005219**	0.008038	0.021296	0.010257	**0.000119**	0.000168	0.002499	0.000714
M2	0.014345	0.010257	**0.001855**	0.002298	0.003316	0.001879	**0.000361**	0.000877
M3	0.049497	0.037105	0.034794	**0.022167**	0.038346	0.012247	0.025256	**0.008201**
M4	0.008672	0.007433	**0.005598**	0.006518	**0.000925**	0.002404	0.003753	0.003842
M5	0.152022	0.113836	0.140459	**0.100007**	0.148395	0.100820	0.133217	**0.093216**
M6	0.056014	0.043471	0.050594	**0.038419**	0.053628	**0.040018**	0.043361	0.041668
M7	0.027229	0.021211	0.024952	**0.019256**	0.012103	**0.004418**	0.014214	0.004566
M8	0.038949	0.027765	0.030994	**0.021352**	0.018364	**0.007779**	0.025824	0.008485
M9	0.010064	**0.009071**	0.025743	0.015451	0.003066	0.001300	0.005760	**0.001122**
M10	0.052183	0.045454	**0.042911**	0.043626	0.047571	0.033729	0.038858	**0.032650**
M11	0.023463	0.020239	0.025946	**0.019101**	0.021022	**0.010154**	0.025315	0.010812
M12	**0.034089**	0.035811	0.048286	0.037611	0.008910	0.005016	0.011260	**0.004757**
M13	0.047050	0.040661	0.052122	**0.031847**	0.033815	**0.020224**	0.038620	0.020977
M14	0.065126	**0.043785**	0.066986	0.045752	0.044921	**0.029098**	0.053015	0.031722
M15	0.005597	0.008574	0.035514	**0.022371**	**0.005597**	0.008574	0.035514	0.022371
M16	0.073712	0.070974	0.080689	**0.065076**	0.065496	**0.049189**	0.064059	0.050704
M17	0.108569	0.088024	0.097095	**0.078830**	0.076755	0.064740	0.073507	**0.063844**
M18	0.050216	**0.049516**	0.056337	0.049953	0.032852	**0.029673**	0.032681	0.031185
M19	0.038135	**0.034433**	0.039294	0.035087	0.027261	**0.022225**	0.031473	0.022280
M20	0.083918	**0.080912**	0.090967	0.083591	0.072655	**0.061176**	0.078217	0.063647

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Table 6: Average integrated squared bias ($\overline{ISB}$) based on 100 replications for 20 models with samples size $n = 100$ and $n = 200$.

	$n = 100$				$n = 200$			
	ISB_{vM}	ISB_{vM-JLN}	ISB_{vM-HG}	ISB_{vM-JSH}	ISB_{vM}	ISB_{vM-JLN}	ISB_{vM-HG}	ISB_{vM-JSH}
M1	**3.04082e-05**	6.34841e-05	0.001504	0.000482	**0.000130**	0.000426	0.002935	0.001184
M2	0.003346	0.001899	**0.000185**	0.000800	0.002320	0.001072	**0.000240**	0.000379
M3	0.035586	0.010318	0.023821	**0.006829**	0.032850	0.009040	0.021154	**0.005623**
M4	0.002290	0.001300	0.001273	**0.001134**	0.001808	0.000601	0.000627	**0.000183**
M5	0.151102	**0.102017**	0.135469	0.111047	0.146730	**0.096512**	0.132177	0.098760
M6	0.053433	**0.041933**	0.042452	0.044853	0.054779	**0.037884**	0.044704	0.040683
M7	0.012055	**0.003418**	0.015161	0.003627	0.011890	**0.003048**	0.014729	0.003406
M8	0.019035	0.007261	0.024175	**0.007019**	0.018971	0.006902	0.021794	**0.006061**
M9	0.003343	0.002097	0.003830	**0.001869**	0.003254	0.001306	0.003497	**0.001161**
M10	0.039047	0.028795	0.033483	**0.028344**	0.039227	0.029577	0.034169	**0.029576**
M11	0.019389	**0.007878**	0.024427	0.008800	0.019270	**0.007573**	0.022607	0.007841
M12	0.006522	**0.003136**	0.009393	0.003439	0.006229	**0.002157**	0.010371	0.002493
M13	0.035241	**0.019218**	0.031964	0.020593	0.033787	**0.017780**	0.038108	0.018465
M14	0.043921	**0.027408**	0.050541	0.029725	0.043617	**0.027364**	0.048747	0.028654
M15	0.004455	0.002909	0.004828	**0.001855**	0.004336	0.002906	0.003675	**0.001368**
M16	0.059258	**0.046620**	0.063620	0.048120	0.059188	**0.046421**	0.062583	0.047996
M17	0.076536	**0.063448**	0.073174	0.063749	0.074252	0.063933	0.073700	**0.061833**
M18	0.031513	**0.029350**	0.032692	0.030959	0.031188	**0.027903**	0.030837	0.028397
M19	0.026697	0.022108	0.028112	**0.022010**	0.026548	0.020033	0.027767	**0.019929**
M20	0.068609	**0.057966**	0.071069	0.058526	0.068106	**0.057721**	0.070831	0.058020

Table 7: Estimate parameters with ML method for Time series of flare azimuths and Long-axis orientations of feldspar laths data sets

	Time series of flare azimuths	Long-axis orientations of feldspar laths
$\hat{\mu}_{ML}$	3.050395	3.136083
$\hat{k}_{ML}$	1.343009	1.015225

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