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Objective Bayesian goodness-of-fit tests for the alpha-skew-normal distribution

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Abstract: The family of alpha-skew-normal (ASN) distributions is a flexible class of three-parameter probability models characterized by their location, scale, and shape. The shape parameter governs both asymmetry and uni-bimodality, allowing the distribution to model unimodal or bimodal data with varying degrees of skewness. This paper proposes an objective Bayesian goodness-of-fit test to determine whether a random sample follows an ASN distribution when parameters are unknown. The test statistics are based on empirical distribution function, whose sampling distributions depend solely on the shape parameter. Their prior predictive distributions, serving as null distributions, are obtained by integrating out the shape parameter with respect to a proper approximation of Jeffreys prior, specifically a Cauchy prior, chosen for its analytical tractability. Critical values are estimated via Monte Carlo simulation. A comprehensive simulation study demonstrates that the proposed tests maintain the nominal significance level across various scenarios and exhibit strong power properties against a range of alternative distributions. Finally, the methodology is illustrated through real-data examples, showcasing its practical applicability.

Keywords: alpha-skew-normal distribution, empirical distribution function, goodness-of-fit test, Jeffreys prior, Monte Carlo simulation, prior predictive distribution.

MSC: 60E05, 62G10, 62G30, 62E17

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1 Introduction

The alpha-skew-normal (ASN) distribution, proposed by Elal-Olivero (2010), is an extension of the normal distribution with a shape parameter that regulates skewness differently than the asymmetric normal model of Azzalini (1985); it also captures a uni-bimodality effect, which gives this family of distributions the flexibility to model unimodal or bimodal data sets that have some degree of skewness. The normal distribution is a special case of the ASN family.

Let $\mathbb{R}$ denote the set of real numbers and ϕ the density function of a standard normal random variable. A continuous random variable Z is said to follow an alpha-skew-normal distribution with shape parameter $\alpha \in \mathbb{R}$, denoted by $Z \sim \text{ASN}(\alpha)$, if its probability density function (pdf) is given by

$$f_Z(z;\alpha) = \frac{(1-\alpha z)^2+1}{2+\alpha^2}\phi(z), \quad z \in \mathbb{R}, \tag{1}$$

where α is a parameter that controls both the skewness and the uni-bimodality of the density function. For positive values of α, the distributions are skewed to the right, and for negative values, the distributions are skewed to the left. The ASN distribution is unimodal if $-1.34 < \alpha < 1.34$ and bimodal otherwise. Furthermore, when $\alpha \to \pm\infty$, Z converges in distribution to a bimodal random variable with pdf $y^2\phi(y)$ for $y \in \mathbb{R}$ (Elal-Olivero, 2010).

Let $\mu \in \mathbb{R}$ and $\sigma > 0$. If $Z \sim \text{ASN}(\alpha)$, then the location-scale ASN distribution can be defined as the distribution of the continuous random variable $X = \mu + \sigma Z$, whose probability density function is given by

$$f_X(x;\mu,\sigma,\alpha) = \left[\frac{\left(1-\alpha\left\{(x-\mu)/\sigma\right\}\right)^2+1}{\sigma\left(2+\alpha^2\right)}\right]\phi\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}. \tag{2}$$

A random variable X following an ASN distribution with location, scale and shape parameters μ, σ, and α, respectively, is denoted as $X \sim \text{ASN}(\mu,\sigma,\alpha)$. When $\mu = 0$ and $\sigma = 1$, the pdf (2) simplifies to equation (1). The skewness of X ranges from -0.811 to 0.811, while the kurtosis varies between -1.333 and 0.749. The cumulative distribution function (cdf) of X is given by

$$F_X(x;\mu,\sigma,\alpha) \quad = \quad \Phi\left(\frac{x-\mu}{\sigma}\right) + \alpha\left(\frac{2\sigma-\alpha(x-\mu)}{\sigma(2+\alpha^2)}\right)\phi\left(\frac{x-\mu}{\sigma}\right). \tag{3}$$

In practice, the ASN distribution remains relatively unknown, yet it has played a significant role in the theoretical development of other asymmetric uni-bimodal distributions. Notable examples include the alpha-skew-Laplace (Harandi and Alamatsaz, 2013), alpha-skew-logistic (Chakraborty and Hazarika, 2014), Balakrishnan-alpha-skew-normal (Chakraborty et al., 2014), and alpha-skew-normal *slash* (Gui, 2014) distributions. Additionally, (Louzada et al., 2016) and (Louzada and Ara, 2019) extended the univariate ASN distribution, as defined in equation (1), to its bivariate and multivariate counterparts, respectively.

Let $X_1,\ldots,X_n$ be a random sample of size n from a population with an unknown cdf F. This paper examines the composite goodness-of-fit problem for the $\text{ASN}(\mu,\sigma,\alpha)$ distribution, testing the null hypothesis

$$H_0 : X_1,\ldots,X_n \sim \text{ASN}(\mu,\sigma,\alpha), \quad \text{where } \mu, \sigma \text{ and } \alpha \text{ are unknown}, \tag{4}$$

versus the alternative hypothesis $H_1 : X_1,\ldots,X_n \nsim \text{ASN}(\mu,\sigma,\alpha)$.

Since this problem has not been previously studied, we propose a test procedure based on empirical distribution function (EDF) statistics as a first approach. The null distributions of the test statistics, which determine the critical threshold and depend on the shape parameter (Stephens, 1986), are their prior predictive distributions. These distributions are obtained by integrating out the shape parameter with respect to its prior distribution. To ensure objectivity, Jeffreys rule is used to derive a prior distribution for the shape parameter. However, since the resulting Jeffreys prior is improper, we propose a proper approximation that corresponds to a well-known distribution.

The remainder of this paper is organized as follows. Section 2 presents the method for approximating Jeffreys prior distribution for the shape parameter. Section 3 defines the test statistics and describes the methodology for estimating their prior predictive distributions. Additionally, a table of critical values based on the most relevant sample quantiles from these distributions is provided. Section 4 reports the results of a Monte Carlo simulation study assessing the type I error probability and power of the tests. In Section 5, the proposed tests are applied to two real datasets: one unimodal and the other bimodal. Finally, Section 6 summarizes the main conclusions.

2 Jeffreys prior distribution for the shape parameter α

For a probability model with a one-dimensional parameter θ, the Jeffreys prior distribution of θ is proportional to the square root of its Fisher information of θ.

According to (Elal-Olivero, 2010), the Fisher information of the shape parameter in the ASN distribution with density function (1) is given by

$$\mathscr{I}(\alpha) = \frac{4(\alpha^2 b_2 + 2b_2 - \alpha^2)}{(2+\alpha^2)^2},$$

where

$$b_2 = \mathbb{E}\left[W^2 \frac{(1-\alpha W)^2}{(1-\alpha W)^2 + 1}\right],$$

and W is a standard normal random variable. Consequently, the Jeffreys prior distribution for α, denoted by $\pi_J(\alpha)$, is given by

$$\pi_J(\alpha) \propto \mathscr{I}(\alpha)^{1/2}.$$

The value of b_2 can be computed numerically for each $\alpha \in \mathbb{R}$, enabling the evaluation of $\mathscr{I}(\alpha)$ and the construction of the corresponding plot of $\pi_J(\alpha)$. The resulting graph is shown in Figure 1

Figure 1 reveals that $\pi_J(\alpha)$ exhibits similarities to well-known distributions, such as the Cauchy, generalized double Pareto (Armagan et al., 2013), and Laplace (or double exponential) distributions, each centered at zero. Since $\pi_J(\alpha)$ is improper, we approximate it using one of these three proper and computationally efficient distributions. To determine the optimal scale parameter λ for the chosen approximation, we minimize the Kullback-Leibler divergence between the normalized Jeffreys prior and each candidate distribution.

Since $\pi_J(\alpha)$ is improper, we consider its normalized version

$$p_J(\alpha) = \frac{\mathscr{I}(\alpha)^{1/2}}{C(\xi)},$$

where

$$C(\xi) = \lim_{\xi\to\infty} \int_{-\xi}^{\xi} \mathscr{I}(\alpha)^{1/2} \quad d\alpha.$$

Let $g_J(\alpha)$ denote the proposed Cauchy approximation given by

$$g_J(\alpha) = \frac{1}{\pi\lambda}\left(1 + \frac{\alpha^2}{\lambda^2}\right)^{-1}.$$

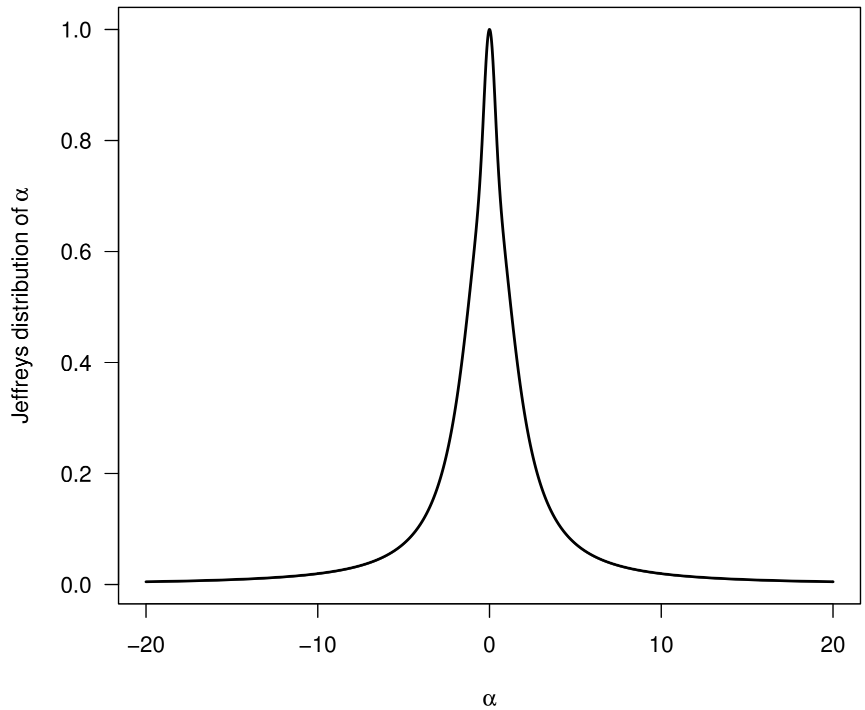

Figure 1: Jeffreys prior distribution for $\alpha \in [-20, 20]$, obtained via numerical integration.

The Kullback-Leibler divergence between $p_J(\alpha)$ and $g_J(\alpha)$ is given by

$$D_{\mathrm{KL}}(p_J, g_J) = \lim_{\xi \to \infty} \left[k(\xi) + \log \lambda + \int_{-\xi}^{\xi} \log \left(1 + \frac{\alpha^2}{\lambda^2} \right) \frac{\mathscr{I}(\alpha)^{1/2}}{C(\xi)} d\alpha \right], \tag{5}$$

where $k(\xi)$ is a term independent of λ. The first-order derivative of above equation with respect to λ is

$$\frac{\partial}{\partial \lambda} D_{\mathrm{KL}}(p_J, g_J) = \lim_{\xi \to \infty} \frac{1}{\lambda} \left[1 - \frac{1}{\lambda^2} \int_{-\xi}^{\xi} 2\alpha^2 \left(1 + \frac{\alpha^2}{\lambda^2} \right)^{-1} \frac{\mathscr{I}(\alpha)^{1/2}}{C(\xi)} d\alpha \right].$$

Setting this expression to zero yields an equation that, when solved numerically, provides the optimal scale parameter λ for the Cauchy approximation. The resulting value, obtained via numerical integration, is $\lambda_{\mathrm{CA}} = 1.48$. Similarly, for the double generalized Pareto and Laplace approximations, the optimal scale parameters are found to be $\lambda_{\mathrm{GPD}} = 1.46$ and $\lambda_{\mathrm{LAP}} = 4.94$, respectively. The resulting distributions are compared with the Jeffreys prior in Figure 2.

Figure 2 shows that the Cauchy distribution provides the best approximation to Jeffreys prior for the shape parameter of the ASN distribution. The generalized double Pareto distribution is also a viable option; however, in the tails, the Cauchy distribution aligns more closely with the normalized Jeffreys prior. Around $\alpha = 0$, both distributions exhibit minimal differences from the Jeffreys prior, suggesting that these small discrepancies should not significantly impact the desirable properties of the tests.

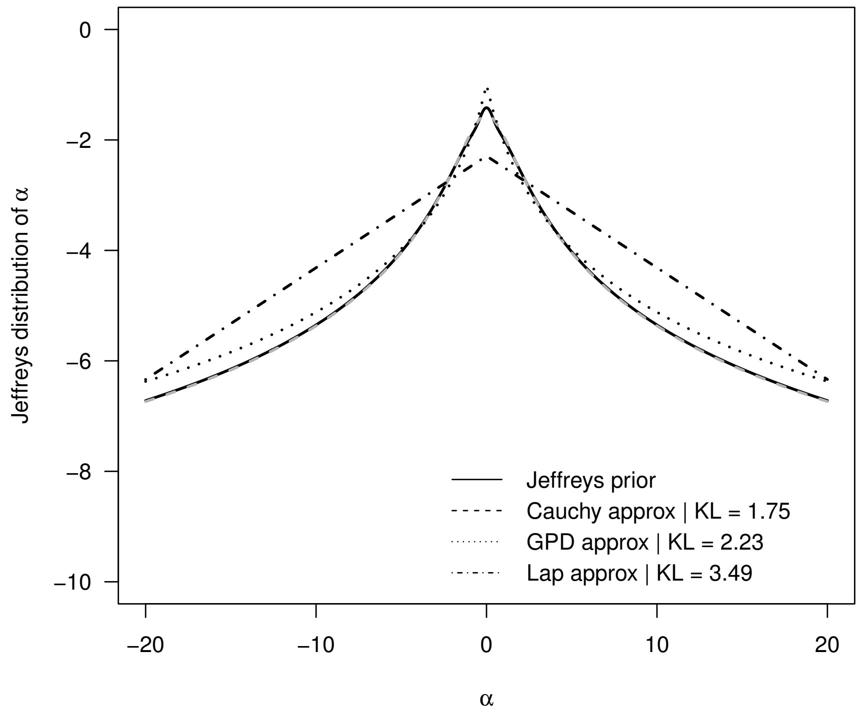

Figure 2: Approximations to the Jeffreys prior distribution for α (in logarithmic scale).

By contrast, the Laplace distribution deviates significantly over nearly the entire range of α. This observation is consistent with the Kullback-Leibler divergence values reported in Figure 2, which indicate that the Cauchy distribution results in the least information loss. Based on this, we propose using a Cauchy distribution centered at zero with a rounded scale parameter of $\lambda = 3/2$ as an approximation of $\pi_J(\alpha)$. This approximation, denoted by $\tilde{\pi}_J(\alpha)$, is given by

$$\tilde{\pi}_J(\alpha) = \frac{1}{\pi(3/2)} \left(1 + \frac{\alpha^2}{(3/2)^2}\right)^{-1}. \tag{6}$$

3 Tests for alpha-skew-normality

In this section, we introduce the test statistics used for evaluating alpha-skew-normality and describe the methodology to estimate their null distributions under the proposed Bayesian framework. We describe the procedure for estimating the quantiles of the sampling distribution of each when the hypothesized distribution is alpha-skew-normal with parameter values estimated from the data using the maximum likelihood method. Furthermore, we summarize the steps necessary to the test the null hypothesis (4) of interest.

3.1 Test statistics based on EDF

The most widely studied goodness-of-fit procedures for a specific family of distributions in the literature are those based on the EDF, a step function denoted by F_n. Given a realization $x_1, \ldots, x_n$ of size n, the EDF is computed as (Stephens, 1986)

$$F_n(x) = \begin{cases} 0 & \text{if } x < x_{(1)}, \\ i/n & \text{if } x_{(i)} \leq x < x_{(i+1)}, \quad i = 1, \ldots, n-1, \\ 1 & \text{if } x \geq x_{(n)}, \end{cases}$$

where $x_{(1)} < \cdots < x_{(n)}$ denote the ordered sample.

A goodness-of-fit test based on F_n assesses whether the sample follows a distribution with an unknown cdf F, quantifying, from a test statistic, the discrepancy between F_n and F, determining whether H_0 should be rejected. Following Stephens (1986), test statistics based on F_n can be categorized into two families: the Kolmogorov–Smirnov family that contains the Kolmogorov–Smirnov statistic D and the Kuiper statistic V, defined as

$$D = \max\left(D^+, D^-\right), \quad V = D^+ + D^-,$$

where

$$D^+ = \sup_x \left\{F_n(x) - F(x)\right\} \quad \text{and} \quad D^- = \sup_x \left\{F(x) - F_n(x)\right\}.$$

The Cramér-von Mises family that contains the Cramér–von Mises statistic W^2, Watson's statistic U^2 and the Anderson–Darling statistic A^2, defined as

$$W^2 = n \int_{\mathbb{R}} \left\{F_n(x) - F(x)\right\}^2 dF(x),$$

$$U^2 = n \int_{\mathbb{R}} \left(F_n(x) - F(x) - \int_{\mathbb{R}} \left\{F_n(x) - F(x)\right\} dF(x)\right)^2 dF(x),$$

$$A^2 = n \int_{\mathbb{R}} \frac{\left\{F_n(x) - F(x)\right\}^2}{F(x)\left\{1 - F(x)\right\}} dF(x).$$

Given an observed sample, these statistics are computed as follows (Stephens, 1974):

$$D = \max\left(D^+, D^-\right), \tag{7}$$

$$V = D^+ + D^-, \tag{8}$$

$$D^+ = \max_i \left(\frac{i}{n} - p_{(i)}\right), \tag{9}$$

$$D^- = \max_i \left(p_{(i)} - \frac{i-1}{n}\right), \tag{10}$$

$$W^2 = \sum_{i=1}^{n} \left(p_{(i)} - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}, \tag{11}$$

$$U^2 = W^2 - n\left(\bar{p} - \frac{1}{2}\right)^2, \tag{12}$$

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log p_{(i)} + \log\left(1 - p_{(n+1-i)}\right)\right], \tag{13}$$

where $p_{(i)} = F(x_{(i)})$ represents the hypothesized cdf values, and $\bar{p}$ is the arithmetic mean of $p_{(1)}, \ldots, p_{(n)}$. For the problem considered in this paper, F corresponds to equation (3). However, since the parameters μ, σ and α are unknown, they are replaced by their maximum likelihood estimates $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\alpha}$. Hence,

$$F = F_X(x; \hat{\mu}, \hat{\sigma}, \hat{\alpha}). \tag{14}$$

When parameter estimates are used, the null distribution of the test statistics depends on the unknown shape parameter (Mateu-Figueras et al., 2009). As a result, the critical values of the tests also depend on the shape parameter. Formally, the critical constant $k_{1-\gamma}$ for a test of significance level $\gamma \in (0, 1)$ satisfies

$$\max_{\alpha} \mathbb{P}\left(T_{\text{EDF}} > k_{1-\gamma} \mid H_0 \text{ is true}\right) = \gamma, \quad \alpha \in \mathbb{R},$$

where T_{EDF} represents any of the aforementioned test statistics.

There is no general approach to solving this issue. Some proposed methods in the literature include: (i) transforming the sample into one from a known distribution, reducing the problem to a setting where standard tests exist, such as normality transformations (Chen and Balakrishnan, 1995), and (ii) using the parametric bootstrap method to estimate the null distribution of the test statistic and approximate critical values via empirical quantiles (Meintanis, 2007).

In this paper, instead, we propose using the prior predictive distribution of T_{EDF}, obtained by integrating out the shape parameter using its prior distribution. In the previous section, we introduced an approximation to the Jeffreys prior for this parameter, which will be used for this purpose.

3.2 Prior predictive distribution of T_{EDF}

If $\pi(\alpha)$ is a proper prior distribution for the shape parameter, the prior predictive distribution of T_{EDF} is given by

$$\pi(t_{\text{EDF}}) = \int_{\mathbb{R}} \pi(t_{\text{EDF}}|\alpha)\pi(\alpha)\, d\alpha, \tag{15}$$

which represents the marginal distribution of the test statistic. The integral in equation (15) cannot be evaluated analytically but can be approximated by Monte Carlo simulation. To the best of our knowledge, the only reference in the literature applying this approach to goodness-of-fit tests is Cabras and Castellanos (2009), in the context of the asymmetric normal distribution introduced by Azzalini (1985), where the prior predictive p-value (Bayarri and Berger, 2000), originally proposed by Box (1980), serves as a foundation.

Bayarri and Berger (2000) suggest specifying a non-informative prior distribution for the noise parameter to ensure an objective analysis. However, a drawback of noninformative prior distributions is that they are often improper, making the prior predictive distribution (15) improper as well, rendering it invalid for practical use. Nevertheless, a proper distribution that closely approximates the Jeffreys prior for the shape parameter is available, namely, the distribution given in equation (6).

The procedure to estimate the distribution (15) of each test statistic consists of the following three steps:

1. Simulate $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(M)} \sim \tilde{\pi}_J(\alpha)$, with $M = 50,000$.

2. Set the sample size n to n and generate a random sample from the distribution ASN(μ, σ, α) with $\mu = 0$, $\sigma = 1$ and $\alpha = \alpha^{(m)}$, for $m = 1, \ldots, M$. The location and scale parameters were set to $\mu = 0$ and $\sigma = 1$ because the distribution of test statistics based on F_n is invariant to changes in location and scale (Stephens, 1986).

3. On every m-th sample:

   (a) Obtain the maximum likelihood estimates of μ, σ and α.

   (b) Calculate the value of T_{EDF}, using the corresponding expression from equations (7), (8), (11), (12) and (13), with $\hat{p}_{(i)}$ given by

$$\hat{p}_{(i)} = \Phi\left(\frac{x_{(i)} - \hat{\mu}}{\hat{\sigma}}\right) + \hat{\alpha}\left(\frac{2\hat{\sigma} - \hat{\alpha}(x_{(i)} - \hat{\mu})}{\hat{\sigma}(2 + \hat{\alpha}^2)}\right)\phi\left(\frac{x_{(i)} - \hat{\mu}}{\hat{\sigma}}\right), \quad i = 1, \ldots, n,$$

   and $x_{(1)}, \ldots, x_{(n)}$ the sample given in step (2) sorted in non-decreasing order.

The proposed procedure generates a set of M values of T_{EDF}, which are used to estimate its null distribution for a given sample size. Similarly, the $100(1-\gamma)\%$ quantiles of its empirical distribution can be obtained to approximate the critical value $k_{1-\gamma}$ for the corresponding test. The quantiles of the statistics D and V were computed with respect to $\sqrt{n}D$ and $\sqrt{n}V$, as these statistics tend to zero as $n \to \infty$ (Lilliefors, 1967).

All computational procedures were implemented in the R software (R Core Team, 2023). Pseudo-random numbers from the ASN(μ, σ, α) distribution were generated using the acceptance-rejection method based on its stochastic representation, as described by Elal-Olivero (2010). Maximum likelihood estimates for the parameters μ, σ and α were obtained using the `optim` function. Pseudo-random numbers from the Cauchy distribution were generated using the `rcauchy` function. Table 1 presents the 90%, 95%, and 99% quantiles of the empirical distribution of the test statistic for different sample sizes.

3.3 Proposed general test procedure

The following procedure is proposed to test the null hypothesis (4) using a random sample $x_1, \ldots, x_n$ of size n, based on one of the test statistics under consideration:

(1) Compute the maximum likelihood estimates of the parameters μ, σ and α for the ASN(μ, σ, α) distribution, denoted as $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\alpha}$.

(2) Calculate the value of the chosen test statistic using the appropriate formula from equations (7), (8), (11), (12) or (13), where $\hat{p}_{(i)}$ is given by

$$\hat{p}_{(i)} = \Phi\left(\frac{x_{(i)} - \hat{\mu}}{\hat{\sigma}}\right) + \hat{\alpha}\left(\frac{2\hat{\sigma} - \hat{\alpha}(x_{(i)} - \hat{\mu})}{\hat{\sigma}(2 + \hat{\alpha}^2)}\right)\phi\left(\frac{x_{(i)} - \hat{\mu}}{\hat{\sigma}}\right), \quad i = 1, \ldots, n,$$

where $x_{(1)}, \ldots, x_{(n)}$ are the data sorted in non-decreasing order.

(3) For a given significance level $\gamma \in (0, 1)$, obtain the $(1-\gamma)$ quantile of the test statistic corresponding to the sample size n from Table 1. If n is not listed in the table, interpolate linearly between the two closest values of n to approximate the quantile. For $n > 500$, use the quantile associated with $n = 500$.

$1-\gamma$	n	$\sqrt{n}D_n$	$\sqrt{n}V_n$	W_n^2	U_n^2	A_n^2
0.90	20	0.7856	1.2521	0.0993	0.0805	0.5842
	30	0.8084	1.2836	0.1034	0.0824	0.6131
	40	0.8225	1.3032	0.1063	0.0849	0.6251
	50	0.8234	1.3196	0.1077	0.0858	0.6257
	75	0.8236	1.3245	0.1089	0.0867	0.6293
	100	0.8241	1.3244	0.1091	0.0869	0.6298
	150	0.8261	1.3289	0.1098	0.0858	0.6303
	200	0.8293	1.3424	0.1101	0.0868	0.6326
	300	0.8323	1.3461	0.1109	0.0875	0.6318
	400	0.8349	1.3444	0.1096	0.0877	0.6394
	500	0.8371	1.3467	0.1105	0.0875	0.6396
0.95	20	0.8597	1.3456	0.1219	0.0987	0.6964
	30	0.8839	1.3791	0.1302	0.1005	0.7382
	40	0.9096	1.3966	0.1386	0.1034	0.7544
	50	0.9043	1.4006	0.1347	0.1043	0.7541
	75	0.9051	1.4212	0.1341	0.1049	0.7565
	100	0.9102	1.4227	0.1356	0.1058	0.7596
	150	0.9132	1.4377	0.1365	0.1056	0.7631
	200	0.9152	1.4427	0.1379	0.1062	0.7645
	300	0.9159	1.4411	0.1369	0.1069	0.7682
	400	0.9174	1.4435	0.1408	0.1067	0.7719
	500	0.9184	1.4446	0.1406	0.1069	0.7801
0.99	20	1.0129	1.5426	0.1752	0.1393	0.9345
	30	1.0585	1.5652	0.1878	0.1426	1.0061
	40	1.0685	1.6062	0.2011	0.1513	1.1189
	50	1.0862	1.6112	0.2097	0.1511	1.1215
	75	1.0917	1.6196	0.2122	0.1518	1.1381
	100	1.0921	1.6333	0.2158	0.1518	1.1475
	150	1.0919	1.6465	0.2162	0.1568	1.1496
	200	1.0922	1.6467	0.2167	0.1573	1.1483
	300	1.0942	1.6465	0.2166	0.1594	1.1468
	400	1.0936	1.6528	0.2193	0.1599	1.1503
	500	1.0963	1.6501	0.2177	0.1594	1.1499

Table 1: $100th(1-\gamma)\%$ quantile of the empirical distribution of each test statistic, for different sample sizes n and different significance levels γ, estimated by $M = 50,000$ simulations.

(4) Reject H_0 at significance level γ if the test statistic computed in step (2) exceeds the quantile obtained in step (3).

An R script (R Core Team, 2023) containing the necessary functions to implement this test procedure is provided in the appendix.

4 Size and power of the tests

To assess whether the actual type I error probability of each test in the proposed procedure matches or does not exceed the specified significance level, and to evaluate the power of the tests against various alternative distributions, a Monte Carlo simulation study was conducted as described below.

4.1 Simulation experiment

To estimate the type I error probability of the tests, first, B samples of size n are simulated from the ASN(μ, σ, α) distribution with parameters $\mu = 0$, $\sigma = 1$ and different values of $\alpha \in [-20, 20]$ as specified in Table 2. The proposed procedure is then applied using the five test statistics under consideration, with the significance level set at $\gamma = 0.05$. Finally, the proportion of times that H_0 is rejected over the B repetitions is computed. The power of the test was estimated in the same manner, using samples from alternative distributions. For this study, a variety of distributions are considered, including symmetric and asymmetric unimodal distributions, asymmetric bimodal distributions, and mixtures of two normal distributions.

1. $t(v)$: Student's t with v degrees of freedom. When $v = 1$ we have the standard Cauchy distribution.
2. $\mathrm{Lap}(0, 1)$: standard Laplace.
3. $\mathrm{Logit}(0, 1)$: standard logistic.
4. $\mathrm{LN}(\mu, \sigma)$: lognormal with location and scale parameters μ and σ, respectively.
5. $\mathrm{Ga}(a, b)$: gamma with shape and scale parameters a and b, respectively
6. $\mathrm{Exp}(1)$: standard exponential.
7. $\chi^2(\nu)$: chi-square with ν degrees of freedom.
8. $\mathrm{Gu}(0, 1)$: standard Gumbel.
9. $\mathrm{SN}(\lambda)$: standard asymmetric normal with shape parameter λ.
10. BASN(α): Balakrishnan alpha-skew-normal with parameter $\alpha \in \mathbb{R}$. Its probability density function is

$$f_X(x; \alpha) = \frac{1}{C(\alpha)} \left[\frac{(1 - \alpha x)^2 + 1}{2 + \alpha^2} \right]^2 \phi(x), \quad x \in \mathbb{R},$$

    where $C(\alpha) = 3 - 4(2 + \alpha^2)^{-1}$. The distribution is unimodal for $-0.96 < \alpha < 0.96$.
11. ASLap(α): alpha-skew-Laplace with parameter $\alpha \in \mathbb{R}$. Its probability density function is

$$f_X(x; \alpha) = \frac{(1 - \alpha x)^2 + 1}{4(1 + \alpha^2)} f_{\mathrm{LP}}(x), \quad x \in \mathbb{R},$$

    where $f_{\mathrm{LP}}(x)$ denotes the standard Laplace distribution. The distribution is unimodal for $-1 < \alpha < 1$.
12. ASLogit(α): alpha-skew-logistic with parameter $\alpha \in \mathbb{R}$. Its probability density function is

$$f_X(x; \alpha) = \frac{3(1 - \alpha x)^2 + 1}{6 + \pi^2 \alpha^2} f_{\mathrm{LG}}(x), \quad x \in \mathbb{R},$$

    where $f_{\mathrm{LG}}(x)$ denotes the standard logistic distribution. The distribution is unimodal for $-0.8 < \alpha < 0.8$.
13. mixN: mixture of two normal distributions N(μ_A, σ_A) and N(μ_B, σ_B) with mixing parameter $w \in (0, 1)$.

4.2 Results

According to Table 2, the estimated type I error probability of the tests is generally less than or equal to the specified significance level. This indicates that, under the considered configurations of the shape parameter and sample size, all five tests under study maintain a level of γ. Instances where the estimated probability slightly exceeds the nominal significance level may be attributed to the inherent variability of the simulation.

Several key observations from Tables 3 and 4 are as follows:

- The estimated power of the tests behaves consistently, increasing as the sample size increases.

- When the alternative distribution has heavy tails, as in the case of the standard Cauchy or Student's t distribution with 2 degrees of freedom, the tests exhibit high power even for sample sizes as small as $n = 20$. However, as the degrees of freedom of the Student's t distribution increase, the power of the tests decreases.

- For the standard logistic distribution, the tests exhibit very low power, even with sample sizes of $n = 50$ and $n = 100$, indicating that the tests may struggle to distinguish the logistic distribution from the ASN model. In contrast, the standard Laplace distribution is better identified.

- For most of the unimodal asymmetric alternative distributions studied, the tests demonstrate strong discriminatory capability. However, they perform poorly with the Gumbel distribution, where power estimates remain low even for $n = 100$. Similarly, the tests fail to effectively detect the asymmetric normal distribution for the considered values of the asymmetry parameter. However, the power improves as the absolute value of the shape parameter increases. For the standard exponential distribution, the asymmetric Student's t with 1 degree of freedom, and the chi-square distribution with 2 degrees of freedom, the tests show favorable power. However, as the degrees of freedom increase in the last two cases, the power of the tests decreases.

- Among the specified asymmetric bimodal distributions, the BASN distribution is the most effectively discriminated. The tests demonstrate high power for sample sizes of $n = 100$, with power increasing as bimodality and asymmetry intensify.

- When the alternative distribution is a mixture of two normal distributions, the estimated power of the tests increases as the value of one of its parameters increases or as the mixing proportion deviates from $1/2$.

n	α	D	V	W^2	U^2	A^2
20	-20	0.04	0.05	0.04	0.04	0.04
	-15	0.04	0.04	0.05	0.04	0.05
	-10	0.05	0.04	0.05	0.05	0.04
	-5	0.04	0.04	0.03	0.04	0.05
	-3	0.04	0.06	0.03	0.04	0.04
	-1	0.02	0.04	0.02	0.06	0.03
	0	0.02	0.04	0.02	0.04	0.02
	1	0.03	0.04	0.02	0.04	0.02
	3	0.04	0.04	0.03	0.04	0.04
	5	0.05	0.05	0.04	0.05	0.05
	10	0.05	0.06	0.04	0.04	0.05
	15	0.05	0.05	0.05	0.05	0.04
	20	0.04	0.04	0.04	0.04	0.04
50	-20	0.04	0.05	0.04	0.05	0.04
	-15	0.04	0.04	0.05	0.05	0.05
	-10	0.05	0.05	0.05	0.04	0.05
	-5	0.04	0.06	0.04	0.05	0.04
	-3	0.04	0.04	0.04	0.06	0.05
	-1	0.02	0.04	0.03	0.04	0.04
	0	0.01	0.04	0.01	0.03	0.03
	1	0.03	0.04	0.03	0.04	0.04
	3	0.04	0.05	0.04	0.04	0.04
	5	0.05	0.05	0.05	0.04	0.05
	10	0.05	0.05	0.05	0.05	0.05
	15	0.04	0.05	0.04	0.04	0.04
	20	0.04	0.04	0.04	0.05	0.04
100	-20	0.05	0.05	0.03	0.05	0.05
	-15	0.05	0.05	0.05	0.05	0.04
	-10	0.04	0.04	0.04	0.04	0.04
	-5	0.04	0.04	0.03	0.05	0.04
	-3	0.05	0.05	0.03	0.04	0.03
	-1	0.03	0.04	0.03	0.04	0.03
	0	0.01	0.04	0.01	0.04	0.02
	1	0.02	0.04	0.04	0.04	0.03
	3	0.04	0.05	0.03	0.05	0.04
	5	0.05	0.05	0.05	0.05	0.05
	10	0.04	0.05	0.05	0.04	0.05
	15	0.05	0.05	0.05	0.05	0.05
	20	0.04	0.05	0.04	0.05	0.04

Table 2: Estimated probability of type I error of the tests for different values of n and different values of the parameter α, with $\gamma = 0.05$.

	$n = 20$					$n = 50$				
Alternative distribution	D	V	W^2	U^2	A^2	D	V	W^2	U^2	A^2
Symmetric unimodal distributions										
$\text{Ca}(0,1)$	0.73	0.76	0.80	0.75	0.81	0.96	0.95	0.99	0.96	0.99
$t(2)$	0.45	0.46	0.50	0.47	0.51	0.65	0.65	0.67	0.66	0.70
$t(3)$	0.29	0.30	0.35	0.29	0.35	0.40	0.39	0.45	0.40	0.48
$\text{Lap}(0,1)$	0.08	0.08	0.14	0.10	0.15	0.25	0.26	0.30	0.26	0.34
$\text{Logit}(0,1)$	0.04	0.05	0.07	0.05	0.07	0.06	0.08	0.10	0.06	0.09
Asymmetric unimodal distributions										
$\text{LogN}(0,0.5)$	0.18	0.19	0.22	0.18	0.24	0.30	0.31	0.35	0.31	0.39
$\text{Exp}(1)$	0.42	0.41	0.51	0.41	0.54	0.70	0.69	0.77	0.70	0.80
$\text{Ga}(0.5,1)$	0.73	0.72	0.81	0.72	0.84	0.95	0.95	1.00	0.95	1.00
$\text{Ga}(2,1)$	0.23	0.24	0.30	0.25	0.33	0.48	0.51	0.55	0.47	0.58
$\chi^2(2)$	0.40	0.41	0.45	0.42	0.47	0.59	0.61	0.65	0.63	0.66
$\chi^2(4)$	0.08	0.09	0.10	0.08	0.15	0.23	0.24	0.28	0.24	0.32
$\text{Gu}(0,1)$	0.10	0.09	0.12	0.10	0.11	0.14	0.15	0.17	0.14	0.17
$\text{NS}(0.5)$	0.04	0.05	0.06	0.05	0.06	0.04	0.05	0.03	0.05	0.03
$\text{NS}(2.5)$	0.15	0.14	0.19	0.16	0.21	0.29	0.26	0.34	0.25	0.35
Asymmetric bimodal distributions										
$\text{BASN}(2)$	0.27	0.28	0.31	0.26	0.31	0.46	0.47	0.52	0.45	0.53
$\text{BASN}(6)$	0.43	0.41	0.45	0.42	0.48	0.63	0.60	0.64	0.61	0.65
$\text{ASLap}(2)$	0.10	0.12	0.15	0.12	0.14	0.26	0.27	0.32	0.27	0.35
$\text{ASLap}(6)$	0.33	0.33	0.35	0.33	0.36	0.48	0.50	0.53	0.49	0.56
$\text{ASLogit}(2)$	0.17	0.18	0.19	0.16	0.19	0.30	0.29	0.32	0.29	0.31
$\text{ASLogit}(6)$	0.33	0.32	0.36	0.33	0.38	0.46	0.47	0.50	0.46	0.55
$\text{mixN}(0.5,-2,2,1,1)$	0.08	0.12	0.09	0.13	0.10	0.07	0.10	0.07	0.09	0.06
$\text{mixN}(0.3,-2,2,1,1)$	0.13	0.12	0.16	0.13	0.16	0.36	0.35	0.40	0.34	0.42
$\text{mixN}(0.3,-3,3,1,1)$	0.21	0.21	0.24	0.20	0.24	0.51	0.49	0.54	0.50	0.54
$\text{mixN}(0.7,-3,3,2,2)$	0.14	0.14	0.15	0.13	0.16	0.31	0.30	0.36	0.31	0.37

Table 3: Estimated power of tests with different values of n and $\gamma = 0.05$.

	$n = 100$					$n = 300$				
Alternative distribution	D	V	W^2	U^2	A^2	D	V	W^2	U^2	A^2
Symmetric unimodal distributions										
$\text{Ca}(0,1)$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$t(2)$	0.90	0.89	0.95	0.88	0.96	1.00	1.00	1.00	1.00	1.00
$t(3)$	0.61	0.62	0.65	0.61	0.68	0.96	0.95	1.00	0.96	1.00
$\text{Lap}(0,1)$	0.44	0.44	0.50	0.43	0.50	0.92	0.91	0.96	0.92	0.97
$\text{Logit}(0,1)$	0.09	0.09	0.10	0.08	0.10	0.12	0.11	0.15	0.13	0.16
Asymmetric unimodal distributions										
$\text{LogN}(0,0.5)$	0.50	0.52	0.56	0.51	0.61	0.82	0.82	0.85	0.81	0.86
$\text{Exp}(1)$	0.93	0.91	0.96	0.90	0.99	1.00	1.00	1.00	1.00	1.00
$\text{Ga}(0.5,1)$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\text{Ga}(2,1)$	0.71	0.72	0.75	0.70	0.80	1.00	1.00	1.00	0.99	1.00
$\chi^2(2)$	0.91	0.93	0.95	0.92	0.98	1.00	1.00	1.00	1.00	1.00
$\chi^2(4)$	0.50	0.51	0.56	0.52	0.60	0.85	0.87	0.93	0.85	1.00
$\text{Gu}(0,1)$	0.21	0.21	0.25	0.20	0.25	0.40	0.42	0.45	0.40	0.46
$\text{SN}(0.5)$	0.03	0.04	0.02	0.05	0.02	0.01	0.04	0.03	0.03	0.01
$\text{SN}(2.5)$	0.31	0.32	0.35	0.33	0.36	0.55	0.56	0.59	0.55	0.60
Asymmetric bimodal distributions										
$\text{BASN}(2)$	0.76	0.75	0.80	0.74	0.82	1.00	0.98	1.00	0.96	1.00
$\text{BASN}(6)$	0.86	0.87	0.90	0.86	1.00	1.00	1.00	1.00	1.00	1.00
$\text{ASLap}(2)$	0.52	0.51	0.55	0.52	0.60	0.91	0.90	0.95	0.91	1.00
$\text{ASLap}(6)$	0.77	0.76	0.80	0.76	0.82	1.00	1.00	1.00	1.00	1.00
$\text{ASLogit}(2)$	0.43	0.44	0.47	0.42	0.51	0.70	0.72	0.75	0.70	0.80
$\text{ASLogit}(6)$	0.67	0.69	0.72	0.70	0.74	0.95	0.96	1.00	0.97	1.00
$\text{mixN}(0.5,-2,2,1,1)$	0.06	0.05	0.04	0.05	0.05	0.06	0.05	0.05	0.05	0.03
$\text{mixN}(0.3,-2,2,1,1)$	0.48	0.46	0.50	0.46	0.52	0.65	0.61	0.71	0.63	0.72
$\text{mixN}(0.3,-3,3,1,1)$	0.69	0.68	0.72	0.69	0.72	0.80	0.76	0.81	0.78	0.83
$\text{mixN}(0.7,-3,3,2,2)$	0.69	0.68	0.70	0.70	0.71	0.81	0.79	0.82	0.78	0.84

Table 4: Estimated power of tests with different values of n and $\gamma = 0.05$ (continued).

5 Application examples

In this section, we illustrate the application of the proposed tests using two datasets, one unimodal and one bimodal, to demonstrate their practical performance in real-world situations.

5.1 Unimodal data

The variable of interest in this dataset is the relative change in the length of corn seeds under compressive stress, referred to as `strain`. To assess the mechanical damage sustained by corn seeds when subjected to compressive forces, an experiment was conducted in which seeds with varying moisture levels and different endosperm types were compressed until rupture occurred. This dataset, originally analyzed by González-Estrada and Cosmes (2019), contains 90 observations (in millimeters) of `strain` measured in corn seeds with floury endosperm and 8% moisture. The histogram of the data is shown in Figure 3.

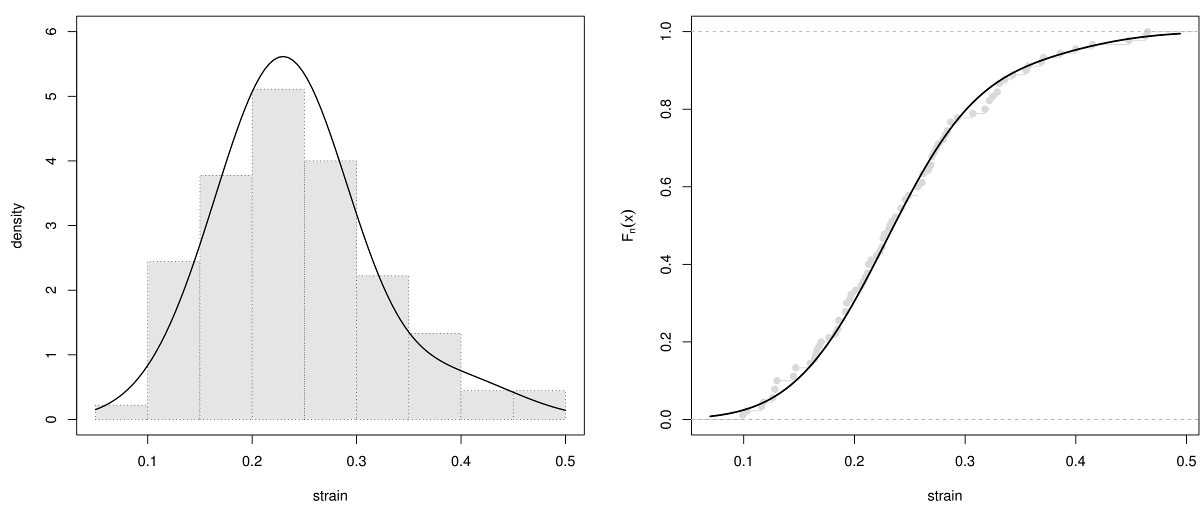

Figure 3: Histogram and empirical distribution function of the `strain` data. The solid line represents the fitted $\mathrm{ASN}(\hat{\mu} = 0.29, \hat{\sigma} = 0.07, \hat{\alpha} = 1.01)$ distribution.

Applying the proposed procedure to this dataset, the test statistic values obtained are

$$\sqrt{n}D_n = 0.5403, \ \ \sqrt{n}V_n = 0.8931, \ \ W_n^2 = 0.0266, \ \ U_n^2 = 0.0264 \ \text{ and } \ A_n^2 = 0.2272.$$

The corresponding 5% significance level quantiles for each test statistic are 0.9081, 1.4247, 0.1338, 0.1051 and 0.7568, respectively. As all computed test statistic values fall below their critical quantiles, the hypothesis of alpha-skew-normality is not rejected. This suggests that the ASN distribution provides a plausible model for describing the stochastic behavior of the data. This conclusion is further supported by Figure 3, which demonstrates a good agreement between the empirical distribution function of the sample and the fitted ASN distribution function.

5.2 Bimodal data

The following example consists of 63 observations of the breaking strength of 1.5 cm-long glass fibers, obtained by the National Physical Laboratory in the United Kingdom. These data can be found, for example, in the text by Jones and Pewsey (2009). The histogram in Figure 4 reveals a small bump, which may indicate the presence of bimodality.

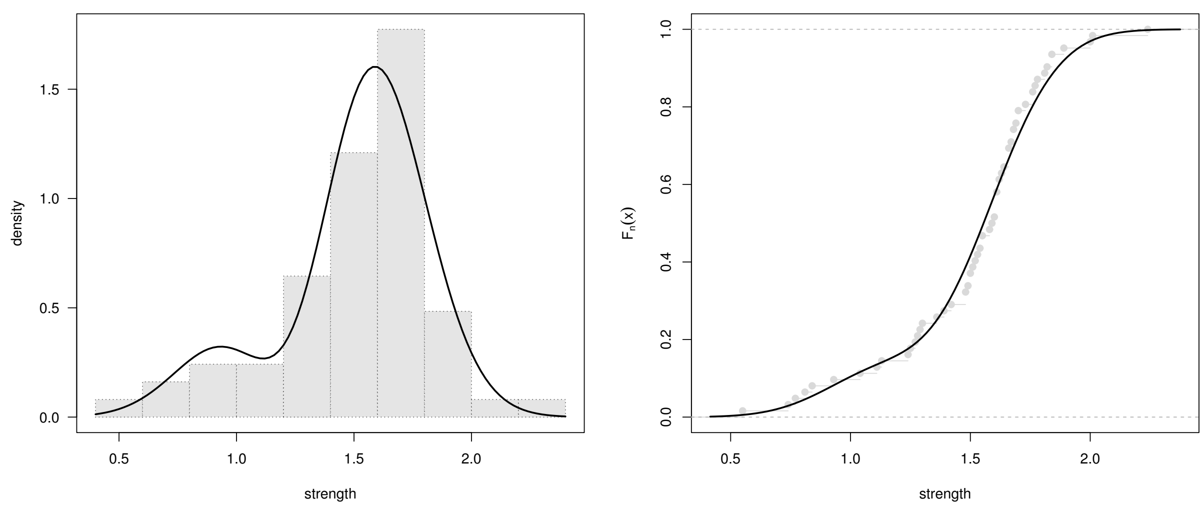

Figure 4: Histogram and empirical distribution function of the `strength` data. The solid line represents the fitted $\mathrm{ASN}(\hat{\mu} = 1.32, \hat{\sigma} = 0.26, \hat{\alpha} = -1.58)$ distribution.

Applying the proposed procedure to the `strength` data, the observed values of the test statistics are

$$\sqrt{n}D_n = 0.7625, \quad \sqrt{n}V_n = 1.2784, \quad W_n^2 = 0.0830, \quad U_n^2 = 0.0788 \text{ and } A_n^2 = 0.4364.$$

At the 5% significance level, the critical quantiles for each test statistic are 0.9046, 1.4114, 0.1344, 0.1041 and 0.7533, respectively. Since all observed test statistics fall below their corresponding quantiles, the results support the alpha-skew-normality hypothesis, suggesting that the ASN distribution provides an appropriate approximation to the frequency distribution of the data. This conclusion is further supported by Figure 4, which shows a good agreement between the empirical distribution function of the data and the fitted ASN distribution function.

6 Conclusions

This paper presents a general testing procedure based on classical goodness-of-fit tests for assessing the validity of any member of the $\mathrm{ASN}(\mu, \sigma, \alpha)$ family of distributions when the parameters μ, σ and α are unknown. The null distribution of each test statistic was approximated using its prior predictive distribution, effectively eliminating dependence on the shape parameter.

The Monte Carlo simulation study provides evidence that the type I error probability of the tests does not exceed the nominal significance level, indicating that the use of the five proposed tests is statistically valid in terms of maintaining the intended significance level. Furthermore, the tests demonstrate strong power against the considered alternative distributions. Among the tests studied, those based on the W^2 and A^2 statistics demonstrate appreciably higher power.

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7 Appendix

```
# Function that calculates the density of the alpha-skew-normal
# distribution of location-scale:
dasnls <- function(x, mu, sigma, alpha){
  w <- (1/sigma)*(x - mu)
  num <- (1 - alpha*w)^2 + 1
  den <- sigma*(2 + alpha^2)
  res <- (num/den)*dnorm(w)
  return(res)
}

# Function that calculates the cumulative density of the
# alpha-skew-normal distribution of location-scale:
pasnls <- function(x ,mu, sigma, alpha){
  w <- (1/sigma)*(x - mu)
  num <- 2*sigma - alpha*(x - mu)
  den <- sigma*(2 + alpha^2)
  res <- pnorm(w) + alpha*dnorm(w)*(num/den)
  return(res)
}

# Function that fits a alpha-skew-normal distribution of location-scale
# to data, using maximum likelihood estimation:
library(moments)
mle_asn <- function(data){
  ll <- function(par){(-1)*sum(log(dasnls(data, mu=par[1], sigma=par[2],
                                          alpha=par[3])))}
  if (skewness(data) > 0){
    init1 <- c(mean(data), sd(data), 1)
    init2 <- c(mean(data), sd(data), 5)
    init3 <- c(mean(data), sd(data), 10)
    opt1 <- optim(init1, ll) ; opt2 <- optim(init2, ll)
    opt3 <- optim(init3, ll)
    indmin <- which.min(c(opt1$value, opt2$value, opt3$value))
```

```
    if(indmin == 1) out <- opt1$par ; if(indmin == 2) out <- opt2$par
    if(indmin == 3) out <- opt3$par
  } else {
    init1 <- c(mean(data), sd(data), -1)
    init2 <- c(mean(data), sd(data), -5)
    init3 <- c(mean(data), sd(data), -10)
    opt1 <- optim(init1, ll) ; opt2 <- optim(init2, ll)
    opt3 <- optim(init3, ll)
    indmin <- which.min(c(opt1$value, opt2$value, opt3$value))
    if(indmin == 1) out <- opt1$par ; if(indmin == 2) out <- opt2$par
    if(indmin == 3) out <- opt3$par
  }
  return(out)
}

# Function that obtains, for a set of data, the value of empirical
# distribution function statistic with respect to the alpha-skew-normal
# distribution:
edfs_asn <- function(data, mu, sigma, alpha){
  n <- length(data)
  xsort <- sort(data)
  pos <- 1:n
  z <- pasnls(xsort, mu, sigma, alpha)
  Dp <- max((pos/n) - z)
  Dm <- max(z - ((pos - 1)/n))
  D <- sqrt(n)*max(Dp, Dm)
  V <- sqrt(n)*(Dp + Dm)
  W2 <- sum((z - ((2*pos - 1)/(2*n)))^2) + (1/(12*n))
  U2 <- W2 - (n*(mean(z) - 0.5)^2)
  aux <- sapply(1:n, function(i){1 - z[n + 1 - i]})
  A2 <- -n - (1/n)*sum((2*pos - 1)*(log(z) + log(aux)))
  return(round(c(D, V, W2, U2, A2), 4))
}

# Function that performs linear interpolation to obtain the constant
# critical:
# tab: table of critical constants.
# n: size of the sample in turn.
interp <- function(tab, n){
  ns <- c(5, 10, 20, 30, 40, 50, 75, 100, 150, 200, 300, 400, 500)
  pos_upp <- 1
  while (n > ns[pos_upp]){
    pos_upp <- pos_upp + 1
  }
  pos_low <- length(ns)
  while (n < ns[pos_low]){
    pos_low <- pos_low - 1
  }
  q_upp <- tab[pos_upp, 3:7]
  q_low <- tab[pos_low, 3:7]
  aux <- ((n - ns[pos_low])/(ns[pos_upp] - ns[pos_low]))
  q_aux <- aux*(q_upp - q_low) + q_low
  return(q_aux)
```

```
}

# Table of critical constants (Table 1):
col1 <- rep(c(0.90, 0.95, 0.99), each=13)
col2 <- rep(ns, times=3)
col3 <- c(0.6911, 0.7384, 0.7856, 0.8084, 0.8225, 0.8234, 0.8236, 0.8241,
          0.8261, 0.8293, 0.8323, 0.8349, 0.8371, 0.7541, 0.8172, 0.8597,
          0.8839, 0.9096, 0.9043, 0.9051, 0.9102, 0.9132, 0.9152, 0.9159,
          0.9174, 0.9184, 0.8833, 0.9578, 1.0129, 1.0585, 1.0685, 1.0862,
          1.0917, 1.0921, 1.0919, 1.0922, 1.0942, 1.0936, 1.0963)
col4 <- c(1.1166, 1.1856, 1.2521, 1.2836, 1.3032, 1.3196, 1.3245, 1.3244,
          1.3289, 1.3424, 1.3461, 1.3444, 1.3467, 1.1976, 1.2721, 1.3456,
          1.3791, 1.3966, 1.4006, 1.4212, 1.4227, 1.4377, 1.4427, 1.4411,
          1.4435, 1.4446, 1.3359, 1.4681, 1.5426, 1.5652, 1.6062, 1.6112,
          1.6196, 1.6333, 1.6465, 1.6467, 1.6465, 1.6528, 1.6501)
col5 <- c(0.0807, 0.0898, 0.0993, 0.1034, 0.1063, 0.1077, 0.1089, 0.1091,
          0.1098, 0.1101, 0.1109, 0.1096, 0.1105, 0.0982, 0.1108, 0.1219,
          0.1302, 0.1386, 0.1347, 0.1341, 0.1356, 0.1365, 0.1379, 0.1369,
          0.1408, 0.1406, 0.1417, 0.1674, 0.1752, 0.1878, 0.2011, 0.2097,
          0.2122, 0.2158, 0.2162, 0.2167, 0.2166, 0.2193, 0.2177)
col6 <- c(0.0718, 0.0756, 0.0805, 0.0824, 0.0849, 0.0858, 0.0867, 0.0869,
          0.0858, 0.0868, 0.0875, 0.0877, 0.0875, 0.0879, 0.0909, 0.0987,
          0.1005, 0.1034, 0.1043, 0.1049, 0.1058, 0.1056, 0.1062, 0.1069,
          0.1067, 0.1069, 0.1215, 0.1308, 0.1393, 0.1426, 0.1513, 0.1511,
          0.1518, 0.1518, 0.1568, 0.1573, 0.1594, 0.1599, 0.1594)
col7 <- c(0.4853, 0.5409, 0.5842, 0.6131, 0.6251, 0.6257, 0.6293, 0.6298,
          0.6303, 0.6326, 0.6318, 0.6394, 0.6396, 0.5602, 0.6394, 0.6964,
          0.7382, 0.7544, 0.7541, 0.7565, 0.7596, 0.7631, 0.7645, 0.7682,
          0.7719, 0.7801, 0.7116, 0.9011, 0.9345, 1.0061, 1.1189, 1.1215,
          1.1381, 1.1475, 1.1496, 1.1483, 1.1468, 1.1503, 1.1499)
tabq <- data.frame(col1, col2, col3, col4, col5, col6, col7)
colnames(tabq) <- c("P", "n", "D", "V", "W2", "U2", "A2")

# Function that obtains the critical constant corresponding to each test:
# tab: table of critical constants.
# sig: significance level of the test (0.01, 0.05, 0.10).
# naux: size of the sample in turn.
qtab <- function(tab, sig, naux){
  tab_aux <- tab[which((1 - sig) == tab$P),]
  if (naux %in% tab_aux$n){
    q_aux <- tab_aux[which(naux == tab_aux$n), 3:7]
  } else {
  if (naux > 500){
    q_aux <- tab_aux[which(500 == tab_aux$n), 3:7]
  } else {q_aux <- interp(tab_aux, naux)}
  }
  return(q_aux)
}

# Function that performs, from a set of data, the test for
# alpha-skew-normality:
# sig: significance level of the test (0.01, 0.05, 0.10).
# pv: boolean object that takes the value of TRUE if the
```

```
#     p-value of the test is desired, otherwise the value
$     of FALSE is specified.
# The routine returns the following data:
# edfsobs: observed value of each test statistic.
# quantile: value of the critical constant for each test.
# rejectH0: boolean object that takes the value of TRUE if H0 is rejected,
#           otherwise FALSE means that it is not rejected.
# mles: maximum likelihood estimates for the parameters.
# pvalue: p-value for each test.
edfstest_asn <- function(data, sig, pv){
  naux <- length(data)
  mle <- mle_asn(data)
  tobs <- edfs_asn(data, mle[1], mle[2], mle[3])
  qp <- qtab(tabq, sig, naux)
  test <- tobs > qp
  if (pv == FALSE){
    print(list(edfsobs=tobs, quantile=qp, rejectH0=test, mles=mle))
  }
  if (pv == TRUE){
    M <- 1E3
    aux <- edfsdist_asn(naux, rcauchy(M, 0, 3/2))
    pv <- apply(sapply(1:M, function(j){aux[,j] > tobs}), 1, mean)
    print(list(edfsobs=tobs, quantile=qp, rejectH0=test, mles=mle,
               pvalue=pv))
  }
}
```