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Theoretical Spatial Economics and Spatial Econometrics: Space-and Time-non-convexities galore...(*)

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Abstract

In theoretical spatial economics and econometrics non-convexities pop up all the time. This paper first treats three cases in theoretical spatial economics: industrial complex computation, a physical planning case, and a generalized Weber problem, to go on to three cases in spatial econometrics: multiple regimes, optimal regime selection, and finite automata.

The main objectives of our contribution are (i) to show that there is a very intimate link between spatial economics and spatial econometrics, in fact, they are the two sides of the same coin; (ii)- non-convexities should be considered as one of the main features in the two fields, they are the norm rather than the exception; (iii) both disciplines, spatial economics and spatial econometrics, are well equipped to deal with non-convexities but is important that scholars assume this restriction, which gives a prominent role to discrete mathematics and discrete regime analysis.

Keywords: Spatial Economics; Spatial Econometrics; Non-Convexities; Finite-Regimes.

JEL Classification: C21, C50, R15

AMS Classification: 62F03

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Teoría Económica Espacial y Econometría Espacial: No convexidades en el Espacio y en el Tiempo a porrillo...

Resumen

La falta de convexidad es un problema recurrente tanto en economía y como en econometría espacial. Este trabajo trata, en primer lugar, tres casos en teoría económica espacial: diseño de complejos industriales, un caso de planificación física y un problema generalizado de localización tipo Weber; a continuación se abordan tres problemas asociados en econometría espacial: regímenes múltiples, selección del régimen óptimo y autómatas finitos.

Los principales objetivos de nuestra contribución son (i) mostrar que existe una conexión muy íntima entre la economía espacial y la econometría espacial, de hecho, son las dos caras de la misma moneda; (ii) insistir en que la falta de convexidad es uno de los rasgos característicos de los dos campos, es la norma antes que la excepción; (iii) ambas disciplinas, economía espacial y econometría espacial, disponen de técnicas suficientes para tratar el problema de la ausencia de convexidad pero es importante que los investigadores asuman esta restricción, lo que confiere gran importancia al análisis matemático para conjuntos discretos y al análisis de regímenes discretos

Palabras Clave: Economía Espacial; Econometría Espacial; Non-Convexidad; Regímenes finitos.

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1. Introduction

Convexity assumption does play a central role in microeconomics and in general equilibrium theory (Varian, 1992). Convexity requires that all inputs and outputs are divisible, however most commodities are clearly indivisible, which immediately violates convexity. There is no need to insist that this is a very controversial hypothesis as stressed, for example, by Farrell (1959): '*A glance at the world about us should be enough to convince us that most commodities are to some extent indivisible and that many have large indivisibilities*' (p. 377). This is an obvious feature of real world and for this reason non-convexities have been incorporated in mainstream economics beginning with, for example, Arrow and Frank (1971).

Non-convexities were also key features in some of the most influential textbooks in regional economics, like Alonso (1964) or Isard (1975), and they were very present in the classical spatial econometrics textbook of Paelinck and Klaassen (1979). However, during the last decades the discussion in theoretical and quantitative spatial economics has slipped towards the assumptions of convexity, linearity and continuity. New Economic Geography, for example, although stressing the importance of scale economics, is very silent about indivisibility and non-convexity problems (Krugman et al., 1999). Likewise, the preferred framework in spatial econometrics comes along with simplifying assumptions favoring smoothness and continuity.

However, it must be recognized once again that non-convexities pop up all the time in spatial economics and econometrics (Paelinck, 2004; Griffith and Paelinck, 2011). This is the purpose of our paper. In first place, Section 2, three common problems in theoretical spatial economics are treated, namely, planning an industrial complex, a case in physical planning, and a generalized Weber location problem. Then, Section 3 presents an analysis of three characteristic problems related to non-convexity in spatial econometrics: the multiple regimes specification, the optimal regime selection dilemma and the finite automata algorithm. Conclusions and references follow.

2. Theoretical spatial economics

The cases analyzed hereafter are only a small sample of what is going on in spatial economic behavior; they are however sufficient to show that non-convexities should be generally considered in spatial economic analysis.

2.1 The case of setting up a petrochemical plant, or non-convexities in economic space-time

Activity complexes have to be developed over time, so some dynamics should be introduced; quadratic assignment appear to be a suitable method for solving the dynamic industrial complex problem (DICP).

Suppose there to be three activities to be started over time; the inter-temporal character of the problem derives from the fact that investment funds might be limited inside each time period. The idea is then to exploit the following facts:

a) prices are rising over time, a well-known result for important projects (the "*Chunnel*", or Channel-Tunnel in Western Europe is a much quoted example);

b) the presence of activities on a site generates externalities.

2.1.1. The model.

Let us define:

- x_{it} as a binary variable, meaning that activity i is to be started in time-period t;
- a_{it} as the cost of implementing activity i in period t;
- b_{ijt} as the percentage/100 of savings accruing to activity i due to the presence in period t of activity j.

The three activity-three period program can then be specified as follows

$$Min\ \varphi = (\varphi_1 + \varphi_2 + \varphi_3) \quad [1]$$

$$s.t.$$

$$\left.\begin{array}{l}
\varphi_1 = a_{11}x_{11} + a_{12}x_{12}\left(1 - b_{122}x_{21} - b_{132}x_{31}\right) + a_{13}x_{13}\left[1 - b_{123}\left(x_{21} + x_{22}\right) - b_{133}\left(x_{31} + x_{32}\right)\right] \\
\varphi_2 = a_{21}x_{21} + a_{22}x_{22}\left(1 - b_{212}x_{11} - b_{232}x_{31}\right) + a_{23}x_{23}\left[1 - b_{213}\left(x_{11} + x_{21}\right) - b_{233}\left(x_{31} + x_{32}\right)\right] \\
\varphi_3 = a_{31}x_{31} + a_{32}x_{32}\left(1 - b_{312}x_{11} - b_{322}x_{21}\right) + a_{33}x_{33}\left[1 - b_{313}\left(x_{11} + x_{12}\right) - b_{323}\left(x_{21} + x_{22}\right)\right] \\
\mathbf{Jx} = \boldsymbol{\tau}
\end{array}\right\} \quad [2]$$

where $\mathbf{x}$ is the (1x9) "row vector $\mathbf{x}' = [x_{11}; x_{12}; x_{13}; x_{21}; x_{22}; x_{13}; x_{31}; x_{32}; x_{33};]$, $\mathbf{J}$ is a (18x9) binary matrix, and $\boldsymbol{\tau}$ a (9x1) unit column vector, the latter ones building up the so-called assignment conditions guaranteeing appropriate non-contradictory allocations in space and/or time.

If one writes out the **A**-matrix, of $a_{ij}s$ one will see that its structure resembles that of the MPP (matrix permutation problem) which results from the presence of externalities, in the form of plants that should exist before another plant is started. This chain of restrictions explains that the square sub-matrices of **A** have a lower block-triangular structure, i.e. a recursive structure in terms of sets of parameters. Table 1 hereafter presents the data used for a test.

Table 1

Data for problem [1]

Parameters	*Values*
a_{11}	1
a_{12}	2
a_{13}	3
a_{21}	2
a_{22}	3
a_{23}	4
a_{31}	3
a_{32}	4
a_{33}	5
b_{122}, b_{123}	.3
b_{132}, b_{133}	.2
b_{212}, b_{213}	.2
b_{232}, b_{233}	.4
b_{312}, b_{313}	.4
b_{322}, b_{323}	.1

In the case of Table 1, one obtains the optimal solution $x^{o\prime} = [0; 0; 1; 1; 0; 0; 0; 1; 0]$ or, what is the same, the sequence of actions is: $[x_{13},\ x_{21},\ x_{32}]$ with a value 7.1 for the objective function in [1]. See Appendix for the details of the solution.

2.1.2 *An example.*

Below we use the well-known case of a complex inter-temporal assignment presented by Isard et al. (1959).The petrochemical complex chosen was the chain "ethylene-ethylene glycol-dacron polymer-dacron staple", the oil refinery (index 0) being already present, so four activities had to be planned over time.

Table 2 hereafter presents the data; some of them, not available, had to be hypothesized, stars marking them; the symbols have the same meaning as above, it being understood that there are now four periods. The first three columns include the original data, only columns four and five having been included in the calculations. Finally, an inflation rate of 10% has been postulated.

Table 2

Data for a petrochemical complex. Isard et al. (1959)

Coefficient	*UI*	*AL*	*SC*	*TI*	*Ex*
a_{11}	2.65	25	.67	22.90	-
a_{22}	4.31	1	.625	4.31	-
a_{33}	3	1*	.67*	2.16*	-
a_{44}	3	1*	.67*	2.16*	-
b_{011},b_{012}, b_{013},b_{014}	-	-	-	-	.1166
b_{122},, b_{123}, b_{124}	-	-	-	-	.0275
b_{212}, b_{213}, b_{214}	-	-	-	-	.0070
b_{232}, b_{233}, b_{234}	-	-	-	-	.0054
b_{322}, b_{323}, b_{324}	-	-	-	-	.0037
b_{342}, b_{343}, b_{344}	-	-	-	-	.0037
b_{432}, b_{433}, b_{434}	-	-	-	-	.0059

UI: Unit investment; AL: Activity level; SC: Economies of scale coefficient; TI: Total investment; Ex: Externalities.

The technical sequence of the petrochemical process obtained from the QAP is x_{11},x_{22}, x_{33}, x_{44}, with an objective function value $\varphi = 30.39$. It should however be remarked that in a different simulation, the first two activities were inverted, which amounted to activity *2* provisionally buying ethylene from outside the complex, while activity *1* was under construction.

2.1.3 *Further specifications*

Many refinements can be introduced in the optimization technique set for in Section 2.1.1.

A first one is the incidence of a possible discounting, though the program presented computed the total expected optimal cost for the complex, i.e. the financial envelope to be provided for. Another possibility is the introduction of financing constraints, possibly in function of the duration of each of the sub-projects.

The technique exposed is sufficiently flexible to accommodate those two refinements, and also other ones that could creep up in special programs; one of them is to program simultaneously more than one complex. How this can be done will be exposed now.

What was implicitly assumed above was that the complex was already given a location. This however, is a most simplifying assumption, as differently composed complexes may have jointly optimal locations. In other words, to each activity index, *i*, a locational index, *j* or *r*, should be added.

The possibility exists indeed that more than one location area could be considered; the program above has to be generalized in the following two ways.

In the first place the externalities mentioned will only occur if the plants concerned have been assigned to the same area. In second lieu, new binary variables have to be defined, assigning activities to areas, this together with the logical constraints that activities can only be assigned to one area or another if they are operated at non-zero levels.

An example of the first point is the following; a term – for industries 1 and 2 – equal to $c_1c_2+d_1d_2$, has to be introduced, indicating that to generate the externalities both industries should belong to the same complex. A logical constraint is that the assignment to an area supposes the overall selection of the corresponding activity.

Table 3 presents the data for three additional activities.

Table 3

Petrochemical complex. Additional parameters

$Ac\backslash Pa$	p^*_i	p_i	p_4a_{1i}	p_5a_{2i}	p_6a_{3i}	w_i	r_i	π_i	x_{io}
4	2	3	-	2	2	2	.1	.2	1
5	3	2	1	-	1	1	.1	.2	1
6	4	2	1	2	-	3	.1	.2	1

p^*_i=prices; p_i =input prices; p_4a_{1i},, p_5a_{2i},, p_6a_{3i}=input costs; w_i = wages; r_i=rents; π_i= profits; x_{io}=levels of production

Moreover the following “cross-inputs” between the two groups of activities have been introduced (table 4); the Table corresponds to a_{ij}s, $i{\neq}j$ of Table 1.

Table 4

Cross-inputs

Plants	*1*	*2*	*3*	*4*	*5*	*6*
1				2		
2					2	
3						2
4			1			
5		3				
6	1					

The results of adding three activities to the ones of section 2.1.1 were as follows: all six were selected, four of them being assigned to one area, the remaining two to the other. In each case, one activity generated a supernumerary profit, with total minimal investment costs for the two areas equal to *10.1553 + 9.5452 = 19.7005*.

One more point should be considered, namely that each area has locational advantages, which can be introduced, for example, via the investment cost. In Table 5 the investment cost for the six complexes depend on the area where the investment takes place. The data that appear in this table refer to the investment coefficients of equation [1] for the six activities considered.

Table.5

Locational advantages

Complex	*Area 1*	*Area 2*
1	5	1
2	3	1
3	1	2
4	2	3
5	3	1
6	4	6

This time again four activities were allocated to area 2, the remaining to area 1, which was also the only one to show a supernumerary profit. Total investment costs went down to *4.5812 + 8.6222 = 13.2004*.

The technique should now be applied to real cases, as is done in the one-complex case. A dynamic version, as presented above, could be set up according to the lines just developed.

2.2 A case in physical planning

Curiously enough this study started with some thoughts being given to solving suudokus.

Suudokus (from the Japanese syllables *suu*, figure, and *doku*, unique) are not just a game, but also a challenge to mathematical programming. Here a mathematical programming solution is presented (Koch, 2005) and an application to physical planning suggested.

A suudoku is a 9x9 grid, filled at the start with a certain number of figures. Table 6 presents an example.

Table 6

A starting suudoku grid

1		4		8	7			6
8	2		5				1	7
				1				
9				5		8	3	2
2			8	7			6	
4	8	5		6	2			
6			7	2	5	1		
	1	2	6	4	8			
7						6	2	

As it is well-known, the problem is to fill out the empty cells, in such a way that *all* the figures from 1 through 9 would be present in each of the nine rows, each of the nine columns, and each of the nine 3x3 squares.

Introduce now for the empty cells of Table 6, 43 variables, a_i. For the present exercise they are ordered along the successive rows. From the start the following facts are known:

1. The figures *absent* in every row, column and square are known (not their position!); also obviously their sums.
2. The possible values each cell can take on; indeed, the possibilities per row, column and square are generally different, so the only possible values are their intersection.

A first idea is to use the above mentioned sums in a linear program, minimizing the sum of the sum of differences between the summed unknowns, per row, column and square, and their known sum, under the constraint that each elementary difference should be non-negative. Given the properties of linear programming an extremal solution (i.e. in terms of integer values) will be obtained, the value of the objective function being zero; but there are multiple solutions, as the rules of the game are not necessarily satisfied.

Next idea is then to switch to binary programming, according to the following specification. Take a row, and one of its elements, a_i; one knows that a_i can only take on, say, k values ($1<k<9$). Let us call them a_{ir} so we can define:

$$a_1 = \sum_{r=1}^{k} x_{ir} a_{ir}$$

$$\text{with } x_{ir} = x_{ir}^2 \tag{3}$$

There are two constraints:

$$\sum_{r=1}^{k} x_{ir} = 1; \forall i \tag{4}$$

as only one value can be taken on. Moreover, over the row, values $\{a_{ir}; i, r=1, .., k\}$ are exclusive, whence one has:

$$\sum_{i \in I}^{k} x_{ir} = 1 \tag{5}$$

Where is the set of indices i for which the values are $a_{ir} = a_{js} = a_{kt} = ... = a$ (being a a certain value).

This leads to the solution presented in Table 7 as the complement to Table 6.

Table 7

Solution to the suudoku problem

	9		2			3	5	
		6		3	4	9		
5	7	8	9		6	2	4	8
	6	7	4		1			
	3	1			9	4		5
			3			7	9	1
	4	9					8	3
3						5	7	9
	5	8	1	9	3			4

Suppose now that the nine numbers correspond to the elements of a set of (urban) amenities; one wants them all to be available inside (square) zones, and along a (square) network of routes or roads. The method exposed above can then be used to allocate the missing ones to zones and routes, given the amenities already present. This simple method forms the basis for a physical urban planning experiment.

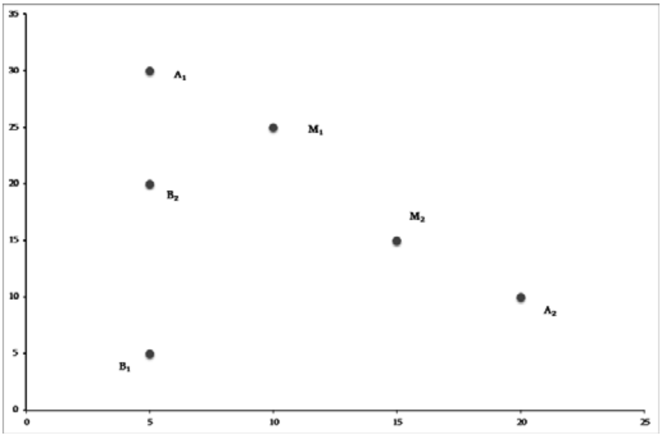
2.3 A generalized Weber problem

The classical Weber problem minimizes total transport cost as a location criterion for an individual firm, F. A number of important features should be added, i.a. multiple purveyors, multiple markets, multiple transportation modes; hereafter those aspects are added.

Two inputs (A, B) with each two purveyors, and two markets are considered; the A_1, A_2, B_1, B_2, M_1, M_2 coordinates are respectively (5,30), (20,10), (5,5), (5,20), (10,25) and (15,15). Moreover the unit transportation costs are more favorable over short distances for mode T_1 than for mode T_2 (respectively 1,2-2,1, 2,3-3,2, 3,4-4,3); technically they were introduced via if-conditions (the cutting values were respectively 10, 15 and 5). Finally the criterion was the minimization of overall transport cost over Manhattan distances, by choosing one A and one B firm, markets M_1 and M_2 having both to be served in proportions of 40% and 60%. The overall situation is depicted in Figure 1 below.

Figure 1

A Weberian location problem: The spatial distribution



The following binary variables are defined

x_1: choice of A_1;

y_1: choice of B_1;

u_1: choice of t_1^A;

v_1: choice of t_1^B;

w_1: choice of t_1^M;

The objective function to be minimized then becomes

$$\begin{aligned}\varphi = a_A\left[u_1t_1^A + (1-u_1)t_2^A\right]\left[x_1d_1^A + (1-x_1)d_2^A\right] + a_B\left[v_1t_1^B + (1-v_1)t_2^B\right]\left[y_1d_1^B + (1-y_1)d_2^B\right] + \\ + q_1\left[w_1t_1^M + (1-w_1)t_2^M\right]d_1^M + q_2\left[w_1t_1^M + (1-w_1)t_2^M\right]d_2^M\end{aligned} \quad [6]$$

In equation [6] symbol *a* stands for the input coefficients, *q* for the market shares, and *d* for the distances. The problem being linear in the distances, a global minimum can be attained. Table 8 shows the solution.

Table 8

Solution for the generalized Weber problem

Element	*Characteristics*	*Values*
F	*Coordinates*	(10,15)
A_1	*Choice*	0
A_2	*Choice*	1
B_1	*Choice*	0
B_2	*Choice*	1
t_1^A	*Choice*	0
t_2^A	*Choice*	1
t_1^B	*Choice*	1
t_2^B	*Choice*	0
t_1^M	*Choice*	0
t_2^M	*Choice*	1
φ	*Objective function*	32

Larger and more complex cases should be investigated, but the method has proven to be operational.

3. Spatial econometrics

Again three cases will be studied, to wit multiple regimes, optimal regimes selection and finite automata. As said in the introduction to section 2, as the economics underlying spatial econometrics is beset with non-convexities, one should expect the latter to turn up in spatial econometrics.

3.1 Multiple regimes

In theoretical and applied physics space-time models have been classical tools of investigation; it comes to mind to try and *appropriately* apply some of them to spatial econometrics. Those space-time representations have since long been expressed in terms of partial differential equations; considering only one space variable, *x,* and time, *t*, a partial differential equation -abbreviated as PDE- for some function *g(x, t)* is a relation of the form:

$$h\left(x,t;g;g_x,g_t;g_{xx},g_{xt},g_{tt};...\right)=0 \qquad [7]$$

where, in general, *h* is a given function of the independent variables, *x* and *t*, of the still unknown function *g*, and of a finite number of its partial derivatives. One well-known member of that family is the wave equation specified as

$$\ddot{f}\left(x,t\right)=\alpha^2 f''\left(x,t\right) \qquad [8]$$

the double dot meaning the second time-derivative (acceleration), the double prime the second *x*-derivative (curvature); α^2 is a strictly positive parameter. Equation [8], as many other commonly used especially in theoretical (non-quantum) physics, is an expression of *local* interaction. However, in spatial economics (the same as in quantum physics) "non-locality" is the rule. In order to express spatial interaction, equation [8] should be generalized to:

$$\ddot{f}(x,t)=\alpha^2\int_{-1}^{+1} w(x,\xi)f''(\xi,t)\,d\xi \qquad [9]$$

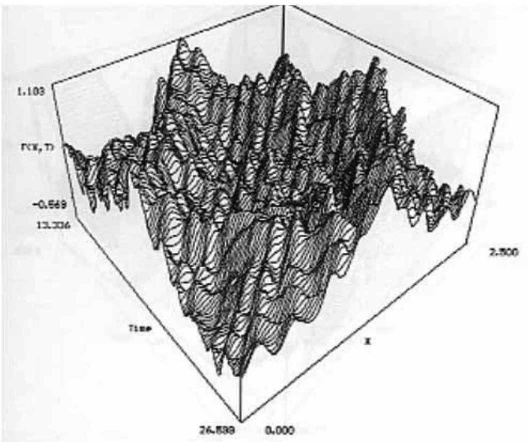
where w(x,ξ) is the so-called "spatial discount function", its convolution with some variable representing a *potential* over a line [-l, +l]. Equation [9] should be called a *potentialized partial differential equation,* PPDE.

Kaashoek and Paelinck (1994, 1996, 1998, 2001) analyzed various features of PPDE's: two-dimensional spatial cases, the effects of varying the potentializing function, and the possibility of controlling the space-time process. Last question results from the fact that the realizations of the process happen to be *chaotic* but, being generated from *exact* equations, they belong to the family of so-called *exact-chaotic* processes.

Figure 2 hereafter pictures one such process taken from Kaashoek and Paelinck, (1998).

Figure 2

A realization of a PPDE



One will notice the presence of sharp "peaks" which have been dubbed "pseudo-solitons", as genuine solitons – as they are called in physics - are in fact infinitely dense local peaks (Dirac functions). However pseudo-solitons, like the true ones, can travel over space, as figure 2 clearly shows; one can imagine figure 2 as a series of peaky waves.

One now has to rewrite equation [10] in a finite difference specification, which results in:

$$\Delta_t^2 f(x,t)=\alpha^2\sum\nolimits_{-1}^{+1} w(x,\xi)\Delta_x^2(\xi,t) \qquad [10]$$

The summation depends on the spatial interaction process selected. As can be seen in Paelinck (2000) potentialized processes can produce very complex spatial patterns.

Model [10] has been applied to the most populated region in France after Ile-de-France, the Rhône-Alpes region (Coutrot et al., 2009). In this case, it was used to analyze the development of knowledge-based industries by means of corresponding employment in activities close to the concept of knowledge-based industries over 3 periods in 39 towns of Rhône-Alpes.

Briefly, the model was first applied to all 39 towns together, but as simulations of first results suggested, there is evidence for the existence of at least two regimes (Griffith and Paelinck, 2011, Chapter 13). Consequently, the equation to be estimated is the following:

$$\begin{aligned}\Delta^2 \ln\left(n_{t0i}\right) &= \lambda\left[a\Delta \ln\left(n_{0i}\right) + b\Delta \ln\left(n_{1i}\right) + c\Delta \ln\left(n_{3i}\right)\right] + \\ &+ (1-\lambda)\left[a^*\Delta \ln\left(n_{0i}\right) + b^*\Delta \ln\left(n_{1i}\right) + c^*\Delta \ln\left(n_{3i}\right)\right]\end{aligned} \quad [11]$$

λ being a binary switching variables, one more instance of non-convexity, typical of spatial economic analysis.

The model has been estimated by minimizing absolute discrepancies with the purpose of neutralizing outliers but, given the result below, any estimator would have done. The results appear in Table 9, (I) referring to model [10], and (II) to model [11]. Note that (I) has been computed from natural values, (II) from natural logarithms.

These results are indeed remarkable from different points of view. First, the regimes are each other's reverse in terms of signs. Second, the fit is almost an interpolation, so all coefficients should be highly significant. Finally, as said, the estimation method problem could be side-stepped; remarkable being also the fact that this happened for a double second-order difference specification. All those results mean that correct specification of a (spatial) econometric model is a central issue.

Apart from these theoretical-econometric considerations, the empirical content of the results should also be considered. In this sense, there is empirical evidence to add robustness to the conclusions, as the (II)-class of Table 9 includes the three main activity centers of the region, to wit Lyons, Grenoble and Saint-Etienne, which moreover have a positive constant and, consequently, a positive autonomous "acceleration".

Table 9

Two regression results

Parameters	*(I)*	*(II)*
A	-0.0041	0.0122
B	-0.0049	0.0072
c	0.0002	-0.0015
a*	-	-0.0059
b*	-	-0.0003
c*	-	0.0008
(Pseudo-)R^2	0.5156	0.9990

The conclusion should be, first that partial difference models do seem to be a very suitable tool to analyze large sets of small spatial units, but second, and most important, that some of the larger units can behave in a different way from the bulge of the set. The latter fact is one of the many instances of multiple regimes, characteristic of spatial econometric practice.

3.2 Optimal regime selection

The preceding section is clearly related to another interesting situation, to wit how to specify a model optimizing the number of regimes. This problem can be compared to optimizing a *p-median* problem with respect to *p*. A possible specification is:

$$
\begin{aligned}
&\mathrm{Min}\textstyle\sum_i \mathrm{P_i} \\
&\text{s.t.} \\
&\left.\begin{aligned} &\textstyle\sum_j \mathrm{SR_j} \le \mathrm{v} \\ &\mathrm{P_j} = \mathrm{P_j^2} \end{aligned}\right\}
\end{aligned}
\qquad [12]
$$

The p_is are the regime selection binary variables as stated in the second restriction. The first restriction must be read in this way: the sum of squared residuals (SR_j) should not exceed some given value *v*, or, alternatively, that the overall (global, pseudo) determination coefficient should exceed *one* minus that value.

This specification has been applied to a linear dynamic model aiming to explain the growth of the Dutch regions (Griffith and Paelinck, 2011, Chapter 11):

$$y_{it} = \theta_1 y_{it-1} + \theta_1 z_{it-1} + \alpha_i + \varepsilon_{it} \qquad [13]$$

y_{it} represent the relative Gross Regional Product (GRP) of the i-th Dutch macro-region in period t whereas z_{it} is the sum of the remaining regions' values. As said the two variables are expressed in shares, for the period 1988-2000, for the case of the four regional aggregates (North, South, East and West) that appear in Figure 4.

Figure 4

The four Dutch macro-regions

The data are included in Table 10 below.

Table 10

Relatives GRP for the four Dutch macro-regions

Year	*North*	*South*	*East*	*West*
1988	10.8094	20.0306	17.3263	51.8337
1989	10.4381	20.1981	17.2407	52.1231
1990	10.4117	20.4517	17.4317	51.7049
1991	10.6066	20.4546	17.5047	51.4341
1992	10.9608	20.4557	17.4602	51.1233
1993	10.6208	20.5621	17.6697	51.1474
1994	10.5830	20.3957	17.8816	51.1397
1995	10.2275	20.6878	17.8719	51.2128
1996	10.0604	20.9624	17.7872	51.1900
1997	10.3111	20.9355	17.7193	51.0341
1998	10.1506	20.7147	17.6959	51.4388
1999	9.7498	20.9352	17.6603	51.6547
2000	9.3353	21.0844	17.8538	51.7265
2001	9.5463	20.9916	17.7386	51.7235

The results appear in Table 11. The first two columns indicate the number of spatial regimes considered and the components of each spatial regime; in the other columns there appear the parameter estimates and a measure of goodness-of-fit (as this was a first exercise, LS was used; see also the concluding remarks).

Table 11

Results for model [13]

Regions	*Number of regimes*	θ_1	θ_2	α_1	*(macro)-*R^2	*Estimation method*
(N,W,S,E)	1	1.0001	0.0000	0.6352	0.5566	LS
(N,W);(S.E)	2	-	-	-	0.5735	LS
(N);(S,E);(W)	3	-	-	-	0.6302	LS
(N)	4	0.8711	0.0137	0.8500	0.6430	LS
(S)		0.8226	0.0522	0.7300		
(E)		0.7965	0.0502	0.7400		
(W)		0.8677	0.1399	0.3400		
(N)*	4	-1.2368	-1.2618	141.1399	0.8870	SDLS
(S)*		-0.2450	0.4712	-47.4387	0.8816	
(E)*		-0.2108	0.1072	-5.54545	0.8074	
(W)*		6.7218	0.6220	-748.6307	0.8894	

*From Table 11.3 in Griffith and Paelinck (2011, Chapter 11).

As is to be expected, the (pseudo-macro)-R^2s (i.e. the global variance ratios) increase with the number of regimes. As to model [12], it renders one regime as optimal as long as the desired R^2 does not exceed 0.5566 (first restriction in 12); between 0.5566 and 0.5735 the model selects two regimes.

However, the specification used is far from appropriate to model shares. The last group of results (starred case appearing in the fourth panel) have been obtained by using a mixed Lotka-Volterra specification with endogenously generated SDLS variables (Griffith and Paelinck, 2011, chapter 11). As noted in the chapter just mentioned (page 183) no extra adding-up constraint had to be introduced, which would not have been the case for model [13] which delivers only positive coefficients (Table 11, third panel). Once more, specification is a central issue in spatial econometrics.

3.3 Finite automata

A finite automaton specification (for a formal definition, see Linz, 1996, p. 2) can be viewed as an "if"-specification; in symbolic terms

$$y : \text{if}\left(\alpha_{xi} + \beta < \gamma_{zi} + \delta; \alpha_{xi} + \beta; \gamma_{zi} + \delta\right) \qquad [14]$$

which reads as follows: if $\alpha_{xi} + \beta < \gamma_{zi} + \delta$, then $\alpha_{xi} + \beta$, else $\gamma_{zi} + \delta$

An important problem indeed is that of the algebraic structure to be given to the model under construction. In Paelinck (2002) the possibilities of model specification based on a so-called "min-algebra" were studied. That algebra, in fact, generalizes the specification of the European FLEUR-model (Ancot and Paelinck, 1983), the latter being based on the idea of a "growth threshold". In a min-algebra, one (or several)

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explanatory terms (variables with their reaction coefficients) *of minimal value* determine the value of the endogenous variable(s). So, instead of considering a (linear or non-linear) combination of endogenous, exogenous or predetermined variables, one will only consider one (or a limited number) of explanatory variables in each equation; for instance, the development of a region could be hampered by the absence of a strategic factor, such as technologically highly trained manpower.

A bad specification of regional models becomes really dramatic when they are used to derive "baskets" of regional policy measures. Indeed, the logic of the chosen algebra produces a linear programming solution with more non-zero decision variables than the number of constraints, whereas under a "classical" algebra would in general produce only that number of non-zero decision variables.

The min-algebra can then be translated into a finite automaton as defined above. To submit such a finite automaton model to an empirical test in a well-documented case, gross regional product figures for the Netherlands have been investigated (Griffith and Paelinck, 2011, chapter 14). They were divided in two macro-regional sets, one for the western provinces (Noord-Holland, Zuid-Holland, Utrecht, the so-called "Rimcity"), the other one comprising the data for the remaining provinces.

The curious thing, at first sight, was the behavior of the growth rate values for the non-Rimcity provinces: whatever the state of the location factors attractiveness, they follow the ups and downs of the Rimcity growth rates. This is completely in line with the fact that the Rimcity is indeed the "motor" of the Dutch economy (Paelinck, 1973, pp. 25-40, especially pp.37-40), imposing its evolutionary rhythm to the other regions. This finding has been confirmed with a Lotka-Volterra finite automaton specification (Griffith and Paelinck, 2011, chapter 13).

4. Conclusions

Though being the curse of the spatial economist, theoretical or applied, and also of the spatial econometrician, non-convexities can be and must be handled. Physical planning or the development of industrial complexes, in space and in time, are typical examples where non-convexities play a fundamental role. However, these problems can be treated by using appropriate instruments, i.e., nonlinear optimization techniques. Weberian classical location problems also admit operational solution by linear or nonlinear programming. Our conclusion is that discrete mathematics should become an important analytical tool in all those fields.

The field of spatial econometrics is also prone to discontinuities and non-convexities. We are not talking exactly about spatial heterogeneity, which may be treated by using, for example, local regressions. It is about spatial regimes as a way of discretization of the space where, i.e., the economic growth model changes according to some restriction or threshold effect. This is a field that merits a bit more attention from the scholars.

One more remark from our study that enlarges that discussion: binary conditions might be replaced by fuzzy ones, in the sense that they are replaced by the closed interval

[0,1]. This means that several regimes might apply to the same spatial unit, a fact that has hardly been recognized by spatial analysts.

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Appendix

The problem of Table 1 has been solved according to the following linearization method for Quadratic Assignment Problems (QAPs). Consider any minimising QAP; all QAPs now being binary bilinear problems.

Write a bilinear term *xy* as:

$$xy \Rightarrow x + y - 1 \qquad [A1]$$

For $x = y = 1$, the value of *xy* equals 1; for *x* or *y* equalling *0*, its value is zero; but for $x = y = 0$ its value is -1.

Now construct

$$f(x,y) = a(x + y - 1) + z \geq 0 \qquad [A2]$$

with *z* being a real non-negative auxiliary variable, added to the {*x,y*] set; for $x=y=0$ in a minimizing problem, condition [A2] will result in a value of *z* equal to *a* in equation [A2]. The assignment conditions mentioned after equations [1] of section 2.1.1 then guarantee that not all of the *z*'s will be zero. The number of additional *z*'s that must be computed can be large, for a general MPP-QAP problem. Being *n* the order of vector **x**, the number of bilinear terms amounts to $n^* = n(n-1)/2$, which can be large but not too large for an LP (e.g. for n = 100, n* = 4950; in the example below, n* = 18).

The full linear program or LP now reads as:

$$\underset{x_i;x_j;z_{ij}}{\text{Min}}\ \varphi = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[a_{ij}\left(x_i + x_j - 1\right) + z_{ij}\right]$$

$$\text{s.t.} \qquad [A3]$$

$$\left.\begin{array}{l} a_{ij}\left(x_i + x_j - 1\right) + z_{ij} \geq 0 \\ x_i; x_j; z_{ij} \geq 0 \\ \mathbf{Jx} = \boldsymbol{\tau} \end{array}\right\} \forall_{i,j}$$