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Tail weight measures for distributions. A new tail weight coefficient

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Abstract

Specific information on the structure of a distribution is contained in its extreme values, which is measured using the concept of the tail weight.

In this paper we analyze the concept of tail of a distribution and propose a new definition for constructing the tail set. Using this set, we define a measure to quantify the tail weight in different distributions, which depends on the distribution and on a cut-off value. Fixing the cut-off value, we define a new tail weight coefficient, where this depends only on the distribution used. Of all these elements we define their version in samples, their main properties and their values in some distributions.

Keywords: Tail of distributions, Tail weight measures, Continuous and discrete distributions.

AMS Classification: 62E10, 62G32, 60E05.

Medidas de peso de colas en distribuciones. Un nuevo coeficiente de peso de colas

Resumen

Los valores extremos de una distribución contienen información importante sobre ella, la cual es cuantificada mediante las medidas de peso de colas.

En este trabajo se analiza el concepto de cola de una distribución y se propone cómo determinar sus elementos. Usando este conjunto, se propone una medida para cuantificar su peso en diferentes distribuciones, la cual dependería de una cierta cota. Finalmente, fijando dicha cota se propone un nuevo coeficiente de peso de colas que depende sólo de la distribución en estudio. De todos estos elementos se propone una versión para muestras, se estudian sus principales propiedades y se calculan sus valores en ciertas distribuciones.

Palabras clave: Cola de distribuciones, Medidas de peso de colas, Distribuciones continuas y discretas.

Clasificación AMS: 62E10, 62G32, 60E05.

1. Introduction

The measures to characterize a distribution provide different information about its structure. The most important of these measures including location, scale, dispersion, symmetry, among others, which are defined and used in most basic statistics books.

Specific information about the shape of a distribution is contained in its extreme values, or tails, where the behavior of these values is measured using the concept of tail weight. We may thus refer to a distribution with substantial mass in the extreme values as having heavy tails or high tail weight, so that its density function tends to 0 slowly. This is in contrast to distributions with short tails or low tail weight. This concept can be considered as a characteristic measure of the distribution. Thus, the tail weight in data or sample of variables can be used in inference to calculate certain parameters of its distribution.

Tail weight is used in different areas such as finance research, regarding value-at-risk or profit (Haas and Pigorsch, 2011) or stock returns (Peiro, 1992), telecommunications network analysis (Heyde and Kou, 2004), among others.

To compare the tail weight of two distributions, the ratio between the density functions of extreme values is usually used. Thus, for example, two Normal distributions with different parameters have the same tail weight, whereas for a Normal distribution $N(\mu,\sigma)$ and a t-Student distribution with v degrees of freedom t_v, the tail weight of t_v is heavier than that of $N(\mu,\sigma)$ for any parameters.

A number of studies propose measures to quantify the tail weight in a distribution. Hoaglin *et al.* (1983) define the *index of tail weight* by comparing the standard normal distribution with some quartiles of the distribution under study. Schuster (1984), Heyde and Kou (2004), and Haas and Pigorsch (2011) use the associated *tail exponent measure*. Other measures can be found in Ruppert (1987), Groeneveld and Meeden (1984), Brys *et al.* (2004), Brys *et al.* (2006) or Wang and Serfling (2005). Two special cases, which will be used later as inspiration, are the *mean excess function*, used in insurance sector (Weron, 2004) and in the study of the life expectancy, and the *indicator of tail length* for samples proposes in Hogg (1974).

The kurtosis coefficient is the most widely-used measure of tail weight. For any distribution F with finite moments, it is defined as:

$$C_k(F) = \frac{\mu_4(F)}{\mu_2^2(F)} \quad [1]$$

where $\mu_r(F) = E[(X - E[X])^r]$, with $X \sim F$, denote the central moments of order r of F. Using this coefficient, greater kurtosis indicates greater tail weight. Thus, it is true that $C_k(N(\mu,\sigma)) = 3$ and $C_k(t_v) = 3 + 6/(v-4)$, with $v > 4$, so that this measure indicates that t_v has a heavier tail weight than $N(\mu,\sigma)$, for all $v > 4$.

Alternatives to kurtosis have been proposed in the literature, for example in Balanda and MacGillirray (1988) and Johnson *et al.* (1994).

All these measures have some drawbacks. Specifically, as been stated in Brys *et al.* (2004), the kurtosis coefficient is, actually, a measure of the mass from the flanks to the center of the distribution, restricted to symmetric distribution, which is not clear that it implies that it is a measure of tail weight.

In this paper, we propose a way to identify the elements in the tail of a distribution and, subsequently, we construct a tail weight measure and a new tail weight coefficient. Thus, we analyze the concept of the tail of a distribution, by proposing the tailset ($Tail$) depending on a cut-off value. Using this set, we propose the tailweight measure (m_{tw}) as the mean of the standardized elements in this set. Finally, fixing a cut-off value, we propose the tailweight coefficient (C_{tw}), which depends only on the distribution used. We study the main properties of these new concepts and determine their values for some distributions. Furthermore, we propose a version for samples of all these concepts, so that these can be used in the inference process.

The rest of the paper is organized as follows. In Section 2 we define the tailset and the tailweight measure. In Section 3, we calculate the new measure for some distributions. The tailweight coefficient is detailed in Section 4, where we also study its main properties and calculate its values for some distributions. Section 5 contains some final remarks.

2 Measure of tailweight

The 'tail' of a distribution refers to the set of extreme values in the distribution. In this section we are going to give a way to decide when a value is in the tail of a distribution. We use this information to define a measure of tail weight.

We suggest that a value is in the tail of a distribution if its standardized absolute value is greater than a certain cut-off value. This very simple idea allows us to decide which values of the distribution are in its tail, where the cut-off value is very important in all these processes, being a controversial point and of debate.

From now on, let F be a univariate distribution and $X \sim F$ with range $\Omega \subseteq IR$, μ its mean and σ^2 its variance. Then, we propose the tailset of F with cut-off value $c \geq 0$ as:

$$Tail(F,c) = \left\{x \in \Omega \,/\, \left|\frac{x-\mu}{\sigma}\right| > c\right\} \quad [2]$$

In some cases, we can consider both right and left tail sets, with these sets denoted $Tail^{+}(F,c)$ and $Tail^{-}(F,c)$, respectively, where:

$$Tail^{+}(F,c) = \left\{x \in \Omega \,/\, \frac{x-\mu}{\sigma} > c\right\} \quad \text{and} \quad Tail^{-}(F,c) = \left\{x \in \Omega \,/\, \frac{x-\mu}{\sigma} < -c\right\} \quad [3]$$

Once we know the tailset of a distribution, we can try to establish the mass or weight of this set in the distribution. To do so, we have to find a measure to provide information about the values in the tail set, that is, we have to use a statistic on the tail set. Following Hogg (1974), who uses the mean in some quartiles, we propose the use of the expected value or mean as measure. Accordingly, we propose to measure the tail weight of a distribution as the mean of the values in the tailset, with respect to a certain cut-off value.

Thus, if we consider $X' = |(X-\mu)/\sigma|$ then we define the tailweight measure of F with cut-off value $c \geq 0$ as:

$$m_{tw}(F,c) = E[X' \,|X' > c] \qquad [4]$$

taking by convention that if *Tail(F,c)* is empty then $m_{tw}(F,c) = 0$, and if it is impossible to calculate the conditional expectation then $m_{tw}(F,c) = \infty$, Furthermore, we denote $m_{tw}^{+}(F,c)$ and $m_{tw}^{-}(F,c)$ as the right and left tailweight measures, respectively, where:

$$m_{tw}^{+}(F,c) = E[X'' \mid X'' > c] \quad \text{and} \quad m_{tw}^{-}(F,c) = E[X'' \mid X'' < -c] \qquad [5]$$

whit $X'' = (X-\mu)/\sigma$.

For these definitions it is very easy to proof certain properties and equivalences that will be used to calculate the tailweight measure in some cases. Thus, it is true that:

1. $m_{tw}(F,c) > c$ except if *Tail*(*F*,*c*) is empty.
2. $m_{tw}(F,c)$ is both location and scale invariant. That is, if $(\alpha X + \beta) \sim G$ for any $\alpha \neq 0, \beta \in IR$ then $m_{tw}(G,c) = m_{tw}(F,c)$.
3. If F is a symmetric distribution then $m_{tw}(F,c) = m_{tw}^{+}(F,c) = m_{tw}^{-}(F,c)$.

Property 1 follows immediately from the definition in [4] since it is the mean of the absolute value from values greater than *c*. Property 2 is true since if $Y = \alpha X + \beta$ and $Y' = \left|(Y - \mu_y)\,/\,\sigma_y\right|$ then $Y' = X'$. Finally, the property 3 follows immediately from the definitions in [5].

On the other hand, the tailweight measure can also be defined as $m_{tw}(F,c) = E[h(X)|X \in A]$ where $h(X) = |(X-\mu)\,/\,\sigma|$ and $A = Tail(F,c)$. Thus, if F is a discrete distribution with probability function P_x then:

$$m_{tw}(F,c) = \frac{\sum_{x \in Tail(F,c)} \left|\frac{x-\mu}{\sigma}\right| P_x(x)}{\sum_{x \in Tail(F,c)} P_x(x)} \qquad [6]$$

while if F is a continuous distribution with density function and distribution function f_x and F_x, respectively, then:

$$m_{tw}(F,c) = \frac{\int_{\mu+c\sigma}^{\infty} \frac{x-\mu}{\sigma} f_x(x)dx - \int_{-\infty}^{\mu-c\sigma} \frac{x-\mu}{\sigma} f_x(x)dx}{1-[F_x(\mu+c\sigma)-F_x(\mu-c\sigma)]} \qquad [7]$$

As a particular case, note that if F is a symmetric distribution with $\mu = 0$ and $\sigma^2 = 1$ then:

$$m_{tw}(F,c) = \frac{\int_{c}^{\infty} x f_x(x)dx}{1-F_x(c)} \qquad [8]$$

The definitions in [2] and [4] can be adapted to samples. Thus, if $x = \{x_1, x_2, \dots, x_n\}$ is a data set or sample of $X \sim F$, where $\bar{x}$ and S_x^2 denote the sample mean and the sample variance of x, respectively, we define the tailset of x with cut-off value $c \geq 0$ as:

$$Tail(x,c) = \left\{ x_i \in x \;/\; \left| \frac{x_i - \bar{x}}{S_x} \right| > c \right\} \qquad [9]$$

Furthermore, using the above set, we define the tailweight measure of x with cut-off value c as:

$$m_{tw}(x,c) = \frac{1}{k} \sum_{x_i \in Tail(x,c)} \left| \frac{x_i - \bar{x}}{S_x} \right| \qquad [10]$$

where k is the cardinal of $Tail(x,c)$ and we assume as convention $m_{tw}(x,c) = 0$ if $k = 0$. This measure is in line with some of the properties established for the tailweight measure on distributions, the most important and useful is that this measure is both location and scale invariant. On the other hand, this is a descriptive measure that can be used to estimate some parameters of F.

3 The tailweight measure in some distributions

In this section, we calculate the tailweight measure in [4] for some distributions. For other distributions, we need to use numerical calculus on a computer once some parameters and the cut-off value c have been set. Using the notation of Evans *et al.* (2000) or Casella and Berger (2002) for distributions with mean μ and variance σ^2, the results are shown below.

- *Normal distribution*. For a $N(\mu,\sigma)$ distribution, it is true that:

  $$m_{tw}(N(\mu,\sigma),c) = \frac{\phi\,(c)}{1 - \Phi\,(c)}$$

  where ϕ and Φ are the density function and distribution function of $N(0,1)$, respectively, and since $\int_c^\infty x\,\phi\,(x)dx = \phi\,(c)$

- *Exponential distribution*. For a $Exp\,(\lambda)$ distribution with $\lambda > 0$, it is true that:

  $$m_{tw}\,(Exp(\lambda),c) = \begin{cases} \dfrac{(1-c)e^{-(1-c)} + (1+c)e^{-(1+c)}}{e^{-(1+c)} - e^{-(1-c)} + 1} & if\ c < 1 \\ 1 + c & if\ c \geq 1 \end{cases}$$

  since its range is $[0,\infty)$ then the left tail set is empty if $c \geq \mu/\sigma = 1$, while in other cases we have values in both left and right tail sets.

- *Pareto distribution*. For a $P(a,s)$ distribution with $a > 0$ and $s > 2$ (it is necessary for exit its variance), it is true that:

$$m_{tw}(P(a,s),c)=\begin{cases}\dfrac{\frac{a^s}{(s-1)\sigma}[(\mu+c\sigma)^{-c}(cs\sigma+\mu)+(\mu-c\sigma)^{-c}(cs\sigma-\mu)]}{1+a^s((\mu+c\sigma)^{-c}-(\mu-c\sigma)^{-c})} & if\ c<1\\ \dfrac{s}{s-1}\left(c+\sqrt{\dfrac{s-2}{s}}\right) & if\ c\geq 1\end{cases}$$

Note that if $s > 2$ then $\mu = sa/(s-1)$ and $\sigma^2 = sa^2/((s-1)^2(s-2))$, so that if $c \geq 1$ there are not values in the left tailset, since $\mu - c\sigma \leq a$ and its range is $[a,\infty)$. On the other hand, note that if $c \geq 1$ then the measure is independent on a.

- *Uniform distribution*. For a $Unif(a,b)$ distribution we obtain:

$$m_{tw}(Unif(a,b),c)=\begin{cases}\dfrac{\sqrt{3}+c}{2} & if\ c<\sqrt{3}\\ 0 & if\ c\geq\sqrt{3}\end{cases}$$

Note that if $c \geq \sqrt{3}$ then $\mu - c\sigma \leq a$ and $\mu + c\sigma \geq b$ then there are not values in either tailset.

- *t-Student distribution*. For a t_v distribution with $v>2$ (it is necessary for exit its variance), where f_v and F_v are its density and distribution functions, respectively, we obtain:

$$m_{tw}(t_v,c)=f_{v-2}(c)\left[1-F_v\left(c\sqrt{\frac{v}{v-2}}\right)\right]^{-1}$$

- *Bernoulli distribution*. For a $B(p)$ distribution with $p \in (0,1)$ and defining $a = 1/(1+c^2)$ then:

  – if $c \leq 1$ then $a \geq 1/2$ so that:

$$m_{tw}(B(p),c)=\begin{cases}\sqrt{\dfrac{1-p}{p}} & if\ p\in(0,1-a]\\ 2\sqrt{p(1-p)} & if\ p\in(1-a,a)\\ \sqrt{\dfrac{p}{1-p}} & if\ p\in[a,1)\end{cases}$$

  – if $c > 1$ then $a < 1/2$ so that:

$$m_{tw}(B(p),c) = \begin{cases} \sqrt{\dfrac{1-p}{p}} & if\ p \in (0,a] \\ 0 & if\ p \in (a,1-a) \\ \sqrt{\dfrac{p}{1-p}} & if\ p \in [1-a,1) \end{cases}$$

- *Laplace distribution*. For a $L(a,b)$ distribution (often known as double-exponential), it is true that:

$$m_{tw}(L(a,b),c) = \frac{1}{\sqrt{2}} + c$$

4 A new tail weight coefficient

The tailweight measure, defined in equation [4], depends on both the distribution and a cut-off value. Thus, if we could eliminate or fix the cut-off value in the tailweight measure we could define a tail weight coefficient for any distribution. Furthermore, this new definition could be extended for samples of distributions in order to use it to estimate certain parameters of the distribution.

We have examine a number of possibilities to compare the tailweight measure in a distribution F with the cut-off value c when c tends to ∞, in order to "eliminate" this value, such as $m_{tw}(F,c) - c$ and $m_{tw}(F,c)/c$. All these studies have not been successful. In short, to use the defined concepts we aim to study how to fix c as a positive real number, although this choice is debatable issue and a controversial point of the study.

To fix an appropriate cut-off value two aspects must be taken into account. First, the cut-off value should be as big as possible, since it collects information about the mass of the extreme values of the distribution. And secondly, since we aim to use it in the practice, if c is an excessively large value then it is possible that we find empty tails in many samples of distributions, so that its use would not be useful. Thus, we have analyzed some standardized distributions, like Normal, t-Student, Exponential and Chi-square, for certain values of their parameters. The density functions of the absolute value of these distributions have small differences and these are confused from values of the variable greater than 1.5. On the other hand, using the behavior of the tailweight measure in the Normal distribution as a model to provide a point of comparison, as it is usually done in statistics, it is true that $m_{tw}(N(\mu,\sigma),c)$ is near 2 if c is greater but close to 1.5 and for values greater than 1.65 the probability of this distribution is greater than 0.9, since 1-Φ(1.65)<0.05. In conclusion, we think an appropriate cut-off value will be in the interval (1.5,1.65), where this value only serves to take a reference point and get a numerical value of the tailweight measure for each distribution.

Taking into account the above, we propose as cut-off value in the practice the value $\hat{c}$ defined as:

$$\hat{c} = 1.5718 \qquad [11]$$

where $m_{tw}(N(\mu,\sigma),\hat{c})$ is very close to 2, with an error less than 10^{-5}.

In this situation, we define the tailweight coefficient for a F distribution using the tailweight measure defined in [4] with cut-off value $\hat{c}$, that is:

$$C_{tw}(F) = m_{tw}(F,\hat{c}) \qquad [12]$$

where, by convention, $C_{tw}(F) = 0$ if $Tail(F,\hat{c})$ is empty and $C_{tw}(F)$ is ∞ if it is not possible to calculate the expectation in $Tail(F,\hat{c})$.

As in the case of the tailweight measure in [10], it is possible to define the tailweight coefficient in samples using the cut-off value $\hat{c} = 1.5718$. Thus, if $x = \{x_1,x_2,\ldots,x_n\}$ is a data set or a sample of $X \sim F$, with $\bar{x}$ and S_x^2 the sample mean and the sample variance in x, respectively, we define the tailweight coefficient in x as:

$$C_{tw}(x) = \frac{1}{k}\sum_{x_i \in Tail(x,\hat{c})} \left|\frac{x_i - \bar{x}}{S_x}\right| \qquad [13]$$

where k is the cardinal of $Tail(x,\hat{c})$ defined in equation [9] and by convection we consider $C_{tw}(x) = 0$ if $k = 0$. This concept is both location and scale invariant and it can be used to estimate some parameters of F.

We have calculated this coefficient for some distributions. In some cases, we have simply replaced the value of the cut-off value in the tailweight measure, while in others we have applied calculus on the computer, using R language (R Core Team, 2017). For most distributions, we have obtained the coefficient for different values of the parameters of the distribution, truncating to two decimal places. Using the notation in distributions of Evans *et al.* (2000) or Casella and Berger (2002), the results obtained are as follows.

- *Normal distribution*. $C_{tw}(N(\mu,\sigma)) = 2$.
- *Exponential distribution*. $C_{tw}(Exp(\lambda)) = 2.57$.
- *Pareto distribution*. For $s > 2$ it is true that:

$$C_{tw}(P(a,s)) = \frac{s}{s-1}\left(1.57 + \sqrt{\frac{s-2}{s}}\right)$$

By way of example, the coefficients for some s are:

s	~ 2	2.1	2.5	3	4	5	8	10	25	∞
$C_{tw}(P(a,s))$	3.14	3.41	3.36	3.22	3.03	2.93	2,78	2.74	2.63	2.57

- *Uniform distribution.* $C_{tw}(\mathrm{Unif}(a, b)) = 1.65$.
- *t-Student distribution*. For $\nu > 2$ the coefficients are:

$$C_{tw}(t_v) = f_{v-2}(1.57)\left[1 - F_v\left(1.57\sqrt{\frac{v}{v-2}}\right)\right]^{-1}$$

By way of example, the coefficients for some degrees of freedom are:

s	~ 2	2.1	3	5	8	10	15	20	30	100	∞
$C_{tw}(t_v)$	3.14	3.03	2.53	2.25	2.13	2.10	2.06	2.04	2.03	2.01	2

- *Laplace distribution*. $C_{tw}(L(a, b)) = 2.27$.
- *Chi-squared distribution*. For a Chi-square distribution χ_n^2, the coefficients for some degrees of freedom n using numerical integration by computer are:

n	1	2	3	5	8	10	20	100	∞
$C_{tw}(x_n^2)$	2.78	2.57	2.47	2.37	2.21	2.15	2.06	2.01	2

- *Log-normal distribution*. For a $LN(\mu,\sigma)$ distribution the coefficient is invariant for the μ parameter and it is near 2 if σ tends to 0. Using numerical integration by computer some values of the coefficient are:

σ	0.05	0.1	0.2	0.5	0.75	1	1.5	2	2.5
$C_{tw}(LN(0,\sigma))$	2.00	2.01	2.07	2.51	2.80	3.11	3.75	4.44	5.21

- *Fisher-Snedecor distribution*. For a $F_{n,m}$, the coefficients for some n and $m>4$ integer values using numerical integration by computer are:

n/m	5	6	7	8	9	10	20	30	50	100
1	3.43	3.32	3.23	3.16	3.11	3.08	2.92	2.87	2.83	2.80
2	3.36	3.22	3.11	3.03	2.97	2.93	2.74	2.68	2.63	2.60
3	3.34	3.18	3.07	2.99	2.92	2.87	2.66	2.59	2.54	2.51
4	3.32	3.17	3.05	2.96	2.89	2.84	2.62	2.55	2.49	2.45
5	3.32	3.16	3.04	2.95	2.88	2.82	2.59	2.51	2.45	2.41
10	3.31	3.14	3.01	2.92	2.85	2.79	2.53	2.42	2.31	2.23
20	3.30	3.14	3.01	2.91	2.83	2.77	2.48	2.33	2.21	2.13
100	3.30	3.13	3.00	2.91	2.83	2.77	2.44	2.27	2.15	2.07

Note that if one of the degrees of freedom, n or m, grows and the other one stays fixed then the coefficient decreases.

- *Gamma distribution*. We calculated the coefficient for some values of parameters in a $\gamma(\lambda,r)$ distribution. We have shown that if $\lambda > 0.1$ or if $\lambda < 0.1$ but $r > 0.5$ then the coefficients are very similar. The results produced using numerical integration by computer for $\lambda > 0.1$ and some values of r are:

$\lambda > 0.1$	r=0.1	0.25	0.5	1	2	5	10	50	∞
$C_{tw}(\gamma(\lambda,r))$	3.58	3.06	2.78	2.57	2.41	2.15	2.06	2.01	2

- *Bernoulli distribution*. For a $B(p)$ distribution, where $a = \frac{1}{1+1.57^2} \simeq 0.29$, we obtain:

$$C_{tw}(B(p)) = \begin{cases} \sqrt{\dfrac{1-p}{p}} & if\ p \in (0,a] \\ 0 & if\ p \in (a, 1-a) \\ \sqrt{\dfrac{p}{1-p}} & if\ p \in [1-a, 1) \end{cases}$$

where $C_{tw}\big(B(p)\big) = C_{tw}\big(B(1-p)\big), C_{tw}\big(B(a)\big) = 1.57$ and $C_{tw}\big(B(0.2)\big) = 2$

- *Beta distribution*. For a $\beta(u,v)$, the coefficient for some parameters are:

$u \setminus v$	0.25	0.6	0.8	1	2	4	8	15	25	50
0.1	1.78	2.51	2.72	2.87	3.21	3.40	3.54	3.58	3.61	3.63
0.5	1.76	1.58	1.76	1.89	2.25	2.49	2.63	2.70	2.73	2.77
0.6	1.89	0.00	1.69	1.82	2.17	2.42	2.56	2.63	2.66	2.70
0.75	2.05	1.65	1.61	1.74	2.09	2.33	2.48	2.56	2.59	2.62
1	2.24	1.82	1.72	1.65	1.99	2.23	2.39	2.47	2.51	2.54
2	2.60	2.17	2.06	1.99	1.79	1.94	2.17	2.29	2.33	2.37
3	2.74	2.33	2.22	2.14	1.86	1.88	2.00	2.11	2.18	2.24
5	2.87	2.47	2.37	2.29	2.02	1.90	1.94	2.00	2.04	2.09
10	2.96	2.59	2.49	2.42	2.23	1.98	1.95	1.96	1.99	2.01
20	3.04	2.65	2.56	2.49	2.32	2.06	1.98	1.97	1.98	1.99
50	3.09	2.70	2.60	2.54	2.37	2.14	2.03	2.00	1.99	1.99

The coefficient in this distribution has a very special behavior. For example, it is true that $C_{tw}\big(\beta(u,v)\big) = C_{tw}\big(\beta(v,u)\big)$ and if one of them is fixed and the other one grows then the value of the coefficient has parabolic form, with the minimum when the parameters are equal. An interesting case is when $u = v$, where $C_{tw}\big(\beta(u,u)\big) = 0$ for $u \leq 0.73$ (empty tails), for u near 0.74 (minimum) $C_{tw}\big(\beta(u,u)\big) = 2.57$ and if u tends to ∞ then $C_{tw}\big(\beta(u,u)\big)$ tends to 2.

We have studied the coefficient in other distributions, using simulation and graphics in a computer, observing that it is very fluctuating but with a certain structure. It is the case of distributions like Binomial, Poisson, Geometric and Negative Binomial.

The tailweight coefficient addresses the identities and convergence between distributions. Thus, for example, $C_{tw}\big(\gamma(1/\lambda, 1)\big) = C_{tw}(Exp(\lambda)) = C_{tw}(Exp(2)) = C_{tw}(\chi_2^2) = 2.75$, and, in convergence, it is true that $C_{tw}\big(F_{n,\infty}\big) = C_{tw}(\chi_n^2)$ and the coefficients for

$t_{\infty,} X^2_{\infty,} \gamma(\lambda, \infty), \beta(\infty, \infty)$, and $LN(\mu, 1/\infty)$ tend to 2, since these distributions converge to a Normal distribution. Similarly, in discrete distributions, where Binomial distribution and Poisson distributions are denoted by $Bi(n,p)$ and $Po(\lambda)$, respectively, it is true that $C_{tw}\big(Bi(1,p)\big) = C_{tw}\big(B(p)\big)$ and as $Bi(\infty,p)$ and $Po(\infty)$ tend to a Normal distribution then their coefficients tend to 2.

Finally, using the tailweight coefficient we can consider an order in distributions. Thus, if we denote $D \prec D'$ when $C_{tw}(D) < C_{tw}(D')$ and $D \sim D'$ if $C_{tw}(D) = C_{tw}(D')$, it is true, for example, that:

$$Unif\ (a,b) \prec N(\mu,\sigma) \prec t_8 \prec \chi^2_8 \prec t_5 \prec \chi^2_5 \prec t_3 \prec \chi^2_2 \sim Exp(\lambda) \prec \chi^2_1 \prec t_{2.1} \qquad [14]$$

5 Final remarks

The tail of a distribution is a widely-used concept in a number of different areas, including finance, telecommunication networks among others. Furthermore, this is an important concept when attempting to analyze, understand and characterize a distribution.

This paper proposes a way to define the tail of a distribution, a measure of tail weight and a new coefficient to measure the tail weight of a distribution. All these elements have been studied and calculated in the main distributions in applied statistics.

Furthermore, we extend these concepts to samples, where they can be used in a process of statistical inference to determine some parameters of a distribution. As an example, using the tailweight coefficient in a sample of t-Student distribution we can determine the degrees of freedom. There is scope for future research within this line of study.

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